மனோன்மணியம் சுந்தரனார் பல்கலைக்கழகம்

Manonmaniam Sundaranar University

Reaccredited with 'A' Grade (CGPA 3.13 Out of 4.0) by NAAC (3rd Cycle)

Tirunelveli - 627 012, Tamilnadu, India.

DIRECTORATE OF DISTANCE

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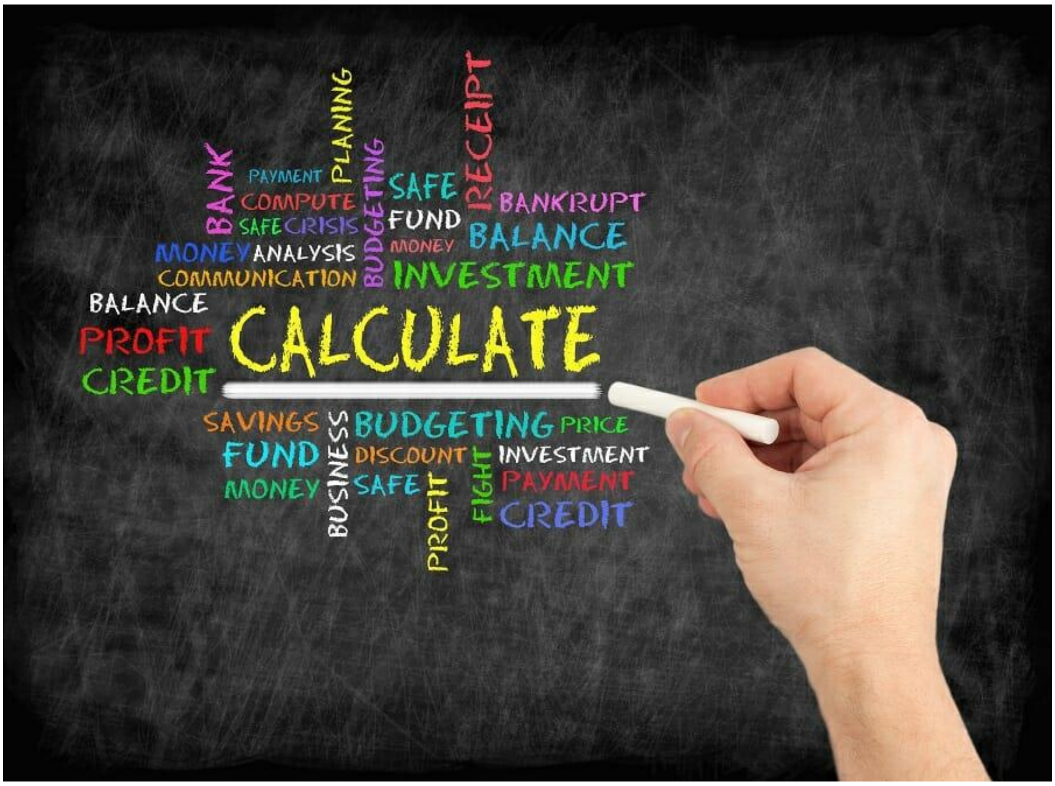
CONTINUING EDUCATION

Quantitative Techniques



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QUANTITATIVE TECHNIQUES

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Objectives:

1. To acquaint the students with the use of quantitative models in decision making
2. To enable the students to learn about formulation of Transportation problem
3. To know about assignment problems
4. To teach the project management and queuing models.
5. To know the evaluation of replacement analysis and simulation.

UNIT I INTRODUCTION AND LINEAR PROGRAMMING:

Operations Research-Quantitative Approach to Decision Making - Nature and Significance of OR in Decision Making-Models in Operations Research- Application Areas of Operation Research- Linear Programming-General Concepts -Definitions - Assumptions in Linear Programming - Limitations in Linear Programming - Applications of Linear Programming - Formulation of LP Problems - Solution Methods : Graphical method (maximization and minimization)- Simplex method (maximization and minimization(Big M Method).

UNIT II TRANSPORTATION

Concepts- Formulation of Transportation Problem- Balanced and Unbalanced ProblemsMinimization and Maximization Problems, Finding IBFS - Northwest Corner Rule(NWCR) , Least Cost Rule (LCR) and Vogel's Approximation Method(VAM) - Optimality Tests - Modified Distribution method (MODI).

UNIT III ASSIGNMENT PROBLEMS

Concepts - Mathematical Formulation of an Assignment Problem - The Assignment Algorithm (Hungarian Assignment method) - Balanced and Unbalanced Assignment ProblemsMinimization and Maximization Problems-Restricted and Reserved routes / choice - Travelling Salesman Problem as an Assignment Problem.

UNIT IV PROJECT MANAGEMENT AND QUEUING MODELS

Introduction- Types of Networks - CPM : Critical Path Method and PERT: Programme Evaluation Review Technique – Basic differences between CPM and PERT - Drawing a network - Obtaining of Critical Path-Time estimates for activities.-Probability of completion of project- Determination of floats (total, free, independent)-Queuing ModelsIntroduction, Concepts, Terminology - General structure of queuing system - Operating Characteristics of Queuing system - Poisson-exponential single server model(finite population)

UNIT V REPLACEMENT ANALYSISAND SIMULATION

Replacement of capital assets – discrete cases when time value of money is not considered and when time value of money is considered.-Replacement of items that fail suddenly -Monte-Carlo method of simulation.

Learning Outcome:

After the completion of the course, the students must be able to:

1. Gain knowledge about formulation of transportation problem
2. Get an outstanding about assignment problems
3. Know about project management and queuing models
4. Gain an understanding about the replacement analysis and simulation

References :

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QUANTITATIVE TECHNIQUES

UNIT – I INTRODUCTION AND LINEAR PROGRAMMING

OPERATIONS RESEARCH

The subject OPERATIONS RESEARCH is a branch of mathematics - specially applied mathematics, used to provide a scientific base for management to take timely and effective decisions to their problems. It tries to avoid the dangers from taking decisions merely by guessing or by using thumb rules. Management is the multidimensional and dynamic concept. It is multidimensional, because management problems and their solutions have consequences in several dimensions, such as human, economic social and political fields. As the manager operates his system in an environment, which will never remain static, hence is dynamic in nature. Hence any manager, while making decisions, consider all aspects in addition to economic aspect, so that his solution should be useful in all aspects. The general approach is to analyse the problem in economic terms and then implement the solution if it does not aggressive or violent to other aspects like human, social and political constraints.

Management may be considered as the process of integrating the efforts of a purposeful group, or organisation, whose members have at least one common goal. You have studied various schools of management in your management science. Most important among them which uses scientific basis for decision making are:

(i) The Decision theory or Decisional Management School and
(ii) The Mathematical or Quantitative Measurement School.

or quantitative methods for making decisions. Quantitative approach to management problems requires that decision problems be defined, analyzed, and solved in a conscious, rational, logical and systematic and scientific manner - based on data, facts, information and logic, and not on mere guess work or thumb rules. Here we use objectively measured decision criteria. Operations research is the body of knowledge, which uses mathematical techniques to solve management problems and make timely optimal decisions. Operations Research is concerned with helping managers and executives to make better decisions. Today's manager is working in a highly competitive and dynamic environment. In present environment, the manager has to deal with systems with complex interrelationship of various factors among them as well as equally complicated dependence of the criterion of effective performance of the system on these factors, conventional methods of decision-making is found very much inadequate. Though the common sense, experience, and commitment of the manager is essential in making decision, we cannot deny the role-played by scientific methods in making optimal decisions. Operations Research uses logical analysis and analytical techniques to study the behaviour of a system in relation to its overall working as resulting from its functionally interconnected constraints, whose parameters are recognized, quantified wherever possible relationships identified to the extent possible and alterative decisions are derived.

Conventional managers were very much worried about that an Operations Research analyst replace them as a decision maker, but immediately they could appreciated him due to his mathematical and logical knowledge, which he applies while making decisions. But operations research analyst list out alternative solutions and their consequences to ease manager's work of decision making. Operations research gives rationality to decision-making with clear view of possible consequences.

The scope of quantitative methods is very broad. They are applied in defining the problems and getting solutions of various organisatons like, business, Government organisations, profit making units and non-profit units and service units. They can be applied to variety of problems like deciding plant location, Inventory control, Replacement problems, Production scheduling, return on investment analysis (ROI), Portfolio selection, marketing research and so on. This book, deals with basic models of Operations research and quantitative methods. The students have to go through advanced Operations Research books, to understand the scope of the subject.

Two important aspects of quantitative methods are: (a) Availability of well-structured models and methods in solving the problems, (b) The attitude of search, conducted on a scientific basis, for increased knowledge in the management of organisations.

Therefore, the attitude encompassed in the quantitative approaches is perhaps more important than the specific methods or techniques. It is only by adopting this attitude that the boundaries and application of the quantitative approach can be advanced to include those areas where, at first glance, quantitative data and facts are hard to come by. Hence, quantitative approach has found place in traditional business and as well in social problems, public policy, national, international problems and interpersonal problems. In fact we can say that the application of quantitative techniques is not limited to any area and can be conveniently applied to all walks of life as far as decision-making is concerned. The quantitative approach does not preclude the qualitative or judgemental elements that almost always exert a substantial influence on managerial decision-making. Quite the contrary. In actual practice, the quantitative approach must build upon, be modified by, and continually benefit from the experiences and creative insight of business managers. In fact quantitative approach imposes a special responsibility on the manager. It makes modern manager to cultivate a managerial style that demand conscious, systematic and scientific analysis - and resolution - of decision problems.

In real world problems, we can notice that there exists a relationship among intuition, judgement, science, quantitative attitudes, practices, methods and models. Higher the degree of complexity and the degree of turbulence in the environment, the greater is the importance of the qualitative approach to management. On the other hand, the lower the degree of complexity i.e., simple and well-structured problems, and lesser degree of turbulence in the environment, the greater is the potential of quantitative models. The advancement in quantitative approach to management problems is due to two facts. They are: (a) Research efforts have been and are being directed to discover and develop more efficient tools and techniques to solve decision problems of all types. (b) Through a continuous process of testing new frontiers, attempts have been made to expand the boundaries and application potential of the available techniques.

Quantitative approach is assuming an increasing degree of importance in the theory and practice of management because of the following reasons. (a) Decision problems of modern management are so complex that only a conscious, systematic and scientifically based analysis can yield a realistic fruitful solution. (b) Availability of list of more potential models in solving complex managerial problems. (c) The most important one is that availability of high speed computers to solve large and complex real world problems in less time and at least cost and which help the managers to take timely decision.

One thing we have to remember here is that if managers are to fully utilize the potentials of management science models and computers, then problems will have to be stated in quantitative terms.

As far as the title of the subject is concerned, the terms ‘quantitative approach’, ‘operations research’, ‘management science’, ‘systems analysis’ and ‘systems science’ are often used interchangeably. What ever be the name of the subject, the syllabi and subject matter dealt which will be same. This analog to ‘god is one but the names are different’.

Quantitative approach to decision making

Many a time we speak of the word decision, as if we know much about decision. But what is decision? What it consists of? What are its characteristics? Let us have brief discussion about the word decision, as much of our time we deal with decision-making process in Operations Research.

A decision is the conclusion of a process designed to weigh the relative uses or utilities of a set of alternatives on hand, so that decision maker selects the best alternative which is best to his problem or situation and implement it. Decision Making involves all activities and thinking that are necessary to identify the most optimal or preferred choice among the available alternatives. The basic requirements of decision-making are (i) A set of goals or objectives, (ii) Methods of evaluating alternatives in an objective manner, (iii) A system of choice criteria and a method of projecting the repercussions of alternative choices of courses of action. The evaluation of consequences of each course of action is important due to sequential nature of decisions.

The necessity of making decisions arises because of our existence in the world with various needs and ambitions and goals, whose resources are limited and some times scarce. Every one of us competes to use these resources to fulfill our goals. Our needs can be biological, physical, financial, social, ego or higher-level self-actualisation needs. One peculiar characteristics of decision-making is the inherent conflict that desists among various goals relevant to any decision situation (for example, a student thinking of study and get first division and at the same time have youth hood enjoyment without attending classes, OR a man wants to have lot of leisure in his life at the same time earn more etc.). The process of decision-making consists of two phases. The first phase consists of formulation of goals and objectives, enumeration of environmental constraints, identification and evaluation of alternatives. The second stage deals with selection of optimal course of action for a given set of constraints. In Operations Research, we are concerned with how to choose optimal strategy under specified set of assumptions, including all available strategies and their associated payoffs.

Objective of Operations Research

"The objective of Operations Research is to provide a scientific basis to the decision maker for solving the problems involving the interaction of various components of an organization by employing a team of scientists from various disciplines, all working together for finding a solution which is in the best interest of the organisation as a whole. The best solution thus obtained is known as optimal decision".

Definition of Operation Research

Any subject matter when defined to explain what exactly it is, we may find one definition. Always a definition explains what that particular subject matter is. Say for example, if a question is asked what is Boyel's law, we have a single definition to explain the same, irrespective of the language in which it is defined.

(a) Operations Research is the art of winning wars without actually fighting. - Aurther Clarke.
(b) Operations Research is the art of giving bad answers to problems where otherwise worse answers are given. - T.L. Satty.
(c) Operations Research is Research into Operations. - J. Steinhardt.
(d) Operations Research is defined as Scientific method for providing executive departments a quantitative basis for decisions regarding the operations under their control. - P.M. Morse and G.E. Kimball.
(e) Operations Research is the study of administrative system pursued in the same scientific manner in which system in Physics, Chemistry and Biology are studied in natural sciences.

CHARACTERISTICS OF OPERATIONS RESEARCH

After considering the objective and definitions of Operations Research, now let us try to understand what are the characteristics of Operations Research.

(a) Operations Research is an interdisciplinary team approach. The problems an operations research analyst face is heterogeneous in nature, involving the number of variables and constraints, which are beyond the analytical ability of one person. Hence people from various disciplines are required to understand the operations research problem, who applies their special knowledge acquired through experience to get a better view of cause and effects of the events in the problem and to get a better solution to the problem on hand. This type of team approach will reduce the risk of making wrong decisions.
(b) Operations Research increases the creative ability of the decision maker. Operations Research provides manager mathematical tools, techniques and various models to analyse the problem on hand and to evaluate the outcomes of various alternatives and make an optimal choice. This will definitely helps him in making better and quick decisions. A manager, without the knowledge of these techniques has to make

decisions by thumb rules or by guess work, which may click some times and many a time put him in trouble. Hence, a manager who uses Operations Research techniques will have a better creative ability than a manager who does not use the techniques.

(c) Operations Research is a systems approach. A business or a Government organization or a defense organization may be considered as a system having various sub-systems. The decision made by any sub-system will have its effect on other sub-systems. Say for example, a decision taken by marketing department will have its effect on production department. When dealing with Operations Research problems, one has to consider the entire system, and characteristics or sub- systems, the inter-relationship between sub-systems and then analyse the problem, search for a suitable model and get the solution for the problem. Hence we say Operations Research is a Systems Approach.

SCOPE OF OPERATIONS RESEARCH

The scope aspect of any subject indicates, the limit of application of the subject matter/techniques of the subject to the various fields to solve the variety of the problems. But we have studied in the objective, that the subject Operations Research will give scientific base for the executives to take decisions or to solve the problems of the systems under their control. The system may be business, industry, government or defense. Not only this, but the definitions discussed also gives different versions. This indicates that the techniques of Operations Research may be used to solve any type of problems. The problems may pertain to an individual, group of individuals, business, agriculture, government or defense. Hence, we can say that there is no limit for the application of Operations Research methods and techniques; they may be applied to any type of problems. Let us now discuss some of the fields where Operations Research techniques can be applied to understand how the techniques are useful to solve the problems. In general we can state that whenever there is a problem, simple or complicated, we can use operations research techniques to get best solution.

(i) In Defense Operations

In fact, the subject Operations research is the baby of World War II. To solve war problems, they have applied team approach, and come out with various models such as resource allocation model, transportation model etc. In any war field two or more parties are involved, each having different resources (manpower, ammunition, etc.), different courses of actions (strategies) for application. Every opponent has to guess the resources with the enemy, and his courses of action and accordingly he has to attack the enemy. For this he needs scientific, logical analysis of the problem to get fruitful results. Here one can apply the techniques like Linear Programming, Game theory, and inventory models etc. to win the game. In fact in war filed every situation is a competitive situation. More over each party may have different bases, such as Air force, Navy and Army. The decision taken by one will have its effect on the other. Hence proper co-ordination of the three bases and smooth flow of information is necessary. Here operations research techniques will help the departmental heads to take appropriate decisions.

(ii) In Industry

After the II World War, the, Industrial world faced a depression and to solve the various industrial problems, industrialist tried the models, which were successful in solving their problems. Industrialist learnt that the techniques of operations research can conveniently applied to solve industrial problems. Then onwards, various models have been developed to solve industrial problems. Today the managers have on their hand numerous techniques to solve different types of industrial problems. In fact decision trees, inventory model, Linear Programming model, Transportation model, Sequencing model, Assignment model and replacement models are helpful to the managers to solve various problems, they face in their day to day work. These models are used to minimize the cost of production, increase the productivity and use the available resources carefully and for healthy industrial growth. An industrial manager, with these various models on his hand and a computer to workout the solutions (today various packages are available to solve different industrial problems) quickly and preciously.

(iii) In Planning For Economic Growth

In India we have five year planning for steady economic growth. Every state government has to prepare plans for balanced growth of the state. Various secretaries belonging to different departments has to co-ordinate and plan for steady economic growth. For this all departments can use Operations research techniques for planning purpose. The question like how many engineers, doctors, software people etc. are required in future and what should be their quality to face the then problems etc. can be easily solved.

(iv) In Agriculture

The demand for food products is increasing day by day due to population explosion. But the land available for agriculture is limited. We must find newer ways of increasing agriculture yield. So the selection of land area for agriculture and the seed of food grains for sowing must be meticulously done so that the farmer will not get loss at the same time the users will get what they desire at the desired time and desired cost.

(v) In Traffic control

Due to population explosion, the increase in the number and verities of vehicles, road density is continuously increasing. Especially in peak hours, it will be a headache to control the traffic. Hence proper timing of traffic signaling is necessary. Depending on the flow of commuters, proper signaling time is to be worked out. This can be easily done by the application of queuing theory.

(vi) In Hospitals

Many a time we see very lengthy queues of patient near hospitals and few of them get treatment and rest of them have to go without treatment because of time factor. Some times we have problems non-availability of essential drugs, shortage of ambulances, shortage of beds etc. These problems can be conveniently solved by the application of operations research techniques. The above-discussed problems are few among many problems that can be solved by the application of operation research techniques. This shows that Operations Research has no limit on its scope of application.

Meaning and Necessity Of Operations Research Models

Management deals with reality that is at once complex, dynamic, and multifacet. It is neither possible nor desirable, to consider each and every element of reality before deciding the courses of action. It is impossible because of time available to decide the courses of action and the resources, which are limited in nature. More over in many cases, it will be impossible for a manager to conduct experiment in real environment. For example, if an engineer wants to measure the inflow of water in to a reservoir through a canal, he cannot sit on the banks of canal and conduct experiment to measure flow. He constructs a similar model in laboratory and studies the problem and decides the inflow of water. Hence for many practical problems, a model is necessary. We can define an operations research model as some sort of mathematical or theoretical description of various variables of a system representing some aspects of a problem on some subject of interest or inquiry. The model enables to conduct a number of experiment involving theoretical subjective manipulations to find some optimum solution to the problem on hand.

Let us take a very simple example. Say you have a small child in your house. You want to explain to it what is an elephant. You can say a story about the elephant saying that it has a trunk, large ears, small eyes etc. The child cannot understand or remember anything. But if you draw small drawing of elephant on a paper and show the trunk, ears, eyes and it will grasp very easily the details of elephant. When a circus company comes to your city and take elephants in procession, then the child if observe the procession, it will immediately recognize the elephant. This is the exact use of a model. In your classrooms your teacher will explain various aspects of the subject by drawing neat sketches on the black board. You will understand very easily and when you come across real world system, you can apply what all you learnt in your classroom. Hence through a model, we can explain the aspect of the subject / problem / system. (Here we can say a model is a relationship among specified variables and parameters of the system).

Classification of Models

The models we use in operations research may broadly classified as:

(i) Mathematical and Descriptive models, and (ii) Static and Dynamic Models.

Mathematical and Descriptive Models

(i) Descriptive Model

A descriptive model explains or gives a description of the system giving various variables, constraints and objective of the system or problem. In article 1.8.1 gives the statement of the problem, which is exactly a descriptive model. The drawback of this model is as we go on reading and proceed; it is very difficult to remember about the variables and constraints, in case the problem or description of the system is lengthy one. It is practically impossible to keep on reading, as the manager has to decide the course of action to be taken timely. Hence these models, though necessary to understand the system, have limited use as far as operations research is concerned.

(ii) Mathematical Model

We have identified the variables and constraints and objective in the problem statement and given them mathematical symbols x and y and a model is built in the form of an inequality of $\leq$ type. Objective function is also given. This is exactly a mathematical model, which explains the entire system in mathematical language, and enables the operations research person to proceed towards solution.

Types of Models

Models are also categorized depending on the structure, purpose, nature of environment, behaviour, by method of solution and by use of digital computers.

a) Classification by Structure

i) Iconic Models: These models are scaled version of the actual object. For example a toy of a car is an iconic model of a real car. In your laboratory, you might have seen Internal Combustion Engine models, Boiler models etc. All these are iconic models of actual engine and boiler etc. They explain all the features of the actual object. In fact a globe is an iconic model of the earth. These models may be of enlarged version or reduced version. Big objects are scaled down (reduced version) and small objects, when we want to show the features, it is scaled up to a bigger version. In fact it is a descriptive model giving the description of various aspects of real object. As far as operations research is concerned, is of less use. The advantages of these models: are It is easy to work with an iconic model in some cases, these are easy to construct and these are useful in describing static or dynamic phenomenon at some definite time. The limitations are, we cannot study the changes in the operation of the system. For some type of systems, the model building is very costly. It will be sometimes very difficult to carry out experimental analysis on these models.

ii) Analogue Model: In this model one set of properties are used to represent another set of properties. Say for example, blue colour generally represents water. Whenever we want to show water source on a map it is represented by blue colour. Contour lines on the map is also analog model. Many a time we represent various aspects on graph by defferent colours or different lines all these are analog models. These are also not much used in operations research. The best examples are warehousing problems and layout problems.

iii) Symbolic Models or Mathematical Models: In these models the variables of a problem is represented by mathematical symbols, letters etc. To show the relationships between variables and constraints we use mathematical symbols. Hence these are known as symbolic models or mathematical models. These models are used very much in operations research. Examples of such models are Resource allocation model, Newspaper boy problem, transportation model etc.

b) Classification by utility

Depending on the use of the model or purpose of the model, the models are classified as Descriptive, Predictive and Prescriptive models.

i) Descriptive model: The descriptive model simply explains certain aspects of the problem or situation or a system so that the user can make use for his analysis. It will not give full details and clear picture of the problem for the sake of scientific analysis.

ii) Predictive model: These models basing on the data collected, can predict the approximate results of the situation under question. For example, basing on your performance in the examination and the discussions you have with your friends after the examination and by verification of answers of numerical examples, you can predict your score or results. This is one type of predictive model.

iii) Prescriptive models: We have seen that predictive models predict the approximate results. But if the predictions of these models are successful, then it can be used conveniently to prescribe the courses of action to be taken. In such case we call it as Prescriptive model. Prescriptive models prescribe the courses of action to be taken by the manager to achieve the desired goal.

c) Classification by nature of environment

Depending on the environment in which the problem exists and the decisions are made, and depending on the conditions of variables, the models may be categorized as Deterministic models and Probabilistic models.

i) Deterministic Models: In this model the operations research analyst assumes complete certainty about the values of the variables and the available resources and expects that they do not change during the planning horizon. All these are deterministic models and do not contain the element of uncertainty or probability. The problems we see in Linear Programming, assumes certainty regarding the values of variables and constraints hence the Linear Programming model is a Deterministic model.

ii) Probabilistic or Stochastic Models: In these models, the values of variables, the pay offs of a certain course of action cannot be predicted accurately because of element of probability. It takes into consideration element of risk into consideration. The degree of certainty varies from situation to situation. A good example of this is the sale of insurance policies by Life Insurance Companies to its customers. Here the failure of life is highly probabilistic in nature. The models in which the pattern of events has been compiled in the form of probability distributions are known as Probabilistic or Stochastic Models.

d) Classification depending on the behaviour of the problem variables

Depending on the behaviour of the variables and constraints of the problem they may be classified as Static Models or Dynamic models. (i) Static Models: These models assumes that no changes in the values of variables given in the problem for the given planning horizon due to any change in the environment or conditions of the system. All the values given are independent of the time. Mostly, in static models, one decision is desirable for the given planning period. (ii) Dynamic Models: In these models the values of given variables goes on changing with time or change in environment or change in the conditions of the given system. Generally, the dynamic models then exist a series of interdependent decisions during the planning period.

e) Classification depending on the method of getting the solution

We may use different methods for getting the solution for a given model. Depending on these methods, the models are classified as Analytical Models and Simulation Models.

i) Analytical Models: The given model will have a well-defined mathematical structure and can be solved by the application of mathematical techniques. We see in our discussion that the Resource allocation model, Transportation model, Assignment model, Sequencing model etc. have well defined mathematical structure and can be solved by different mathematical techniques. For example, Resource allocation model can be solved by Graphical method or by Simplex method depending on the number of variables involved in the problem. All models having mathematical structure and can be solved by mathematical methods are known as Analytical Models.

ii) Simulation Models: The meaning of simulation is imitation. These models have mathematical structure but cannot be solved by using mathematical techniques. It needs certain experimental analysis. To study the behaviour of the system, we use random numbers. More complex systems can be studied by simulation. Studying the behaviour of laboratory model, we can evaluate the required values in the system. Only disadvantage of this method is that it does not have general solution method.

Some Important Models (Problems) We Come Across In The Study Of Operations Research

1. Linear Programming Model

This model is used for resource allocation when the resources are limited and there are number of competing candidates for the use of resources. The model may be used to maximise the returns or minimise the costs. Consider the following two situations:

(a) A company which is manufacturing variety of products by using available resources, want to use resources optimally and manufacture different quantities of each type of product, which yield different returns, so as to maximise the returns.

(b) A company manufactures different types of alloys by purchasing the three basic materials and it want to maintain a definite percentage of basic materials in each alloy. The basic materials are to be purchased from the sellers and mix them to produce the desired alloy. This is to be done at minimum cost. Both of them are resource allocation models, the case (a) is maximisation problem and the case (b) is minimisation problem.

(c) Number of factories are manufacturing the same commodities in different capacities and the commodity is sent to various markets for meeting the demands of the consumers, when the cost of transportation is known, the linear programming helps us to formulate a programme to distribute the commodity from factories to markets at minimum cost. The model used is transportation model.

(d) When a company has number of orders on its schedule, which are to be processed on same machines and the processing time, is known, then we have to allocate the jobs or orders to the machines, so as to complete all the jobs in minimum time. This we can solve by using

Assignment model. All the above-discussed models are Linear Programming Models. They can be solved by application of appropriate models, which are linear programming models.

2. Sequencing Model

When a manufacturing firm has some job orders, which can be processed on two or three machines and the processing times of each job on each machine is known, then the problem of processing in a sequence to minimise the cost or time is known as Sequencing model.

3. Waiting Line Model or Queuing Model

A model used for solving a problem where certain service facilities have to provide service to its customers, so as to avoid lengthy waiting line or queue, so that customers will get satisfaction from effective service and idle time of service facilities are minimised is waiting line model or queuing model.

4. Replacement Model

Any capital item, which is continuously used for providing service or for producing the product is subjected to wear and tear due to usage, and its efficiency goes on reducing. This reduction in efficiency can be predicted by the increasing number of breakdowns or reduced productivity. The worn out parts or components are to be replaced to bring the machine back to work. This action is known as maintenance. A time is reached when the maintenance cost becomes very high and the manager feels to replace the old machine by new one. This type of problems known as replacement problems and can be solved by replacement models.

5. Inventory Models

Any manufacturing firm has to maintain stock of materials for its use. This stock of materials, which are maintained in stores, is known as inventory. Inventory is one form of capital or money. The company has to maintain inventory at optimal cost. There are different types of inventory problems, depending the availability and demand pattern of the materials. These can be solved by the application of inventory models. In fact depending on the number of variables, characteristics of variables, and the nature of constraints different models are available.

LINEAR PROGRAMMING MODELS

INTRODUCTION

A model, which is used for optimum allocation of scarce or limited resources to competing products or activities under such assumptions as certainty, linearity, fixed technology, and constant profit per unit, is linear programming.

Linear Programming is one of the most versatile, powerful and useful techniques for making managerial decisions. Linear programming technique may be used for solving broad range of problems arising in business, government, industry, hospitals, libraries, etc. Whenever we want to allocate the available limited resources for various competing activities for achieving our desired objective, the technique that helps us is **LINEAR PROGRAMMING**. As a decision making tool, it has demonstrated its value in various fields such as production, finance, marketing, research and development and personnel management. Determination of optimal product mix (a combination of products, which gives maximum profit), transportation schedules, Assignment problem and many more. In this chapter, let us discuss about various types of linear programming models.

Properties of Linear Programming Model

Any linear programming model (problem) must have the following properties:

- **(a)** The relationship between variables and constraints must be linear.
- **(b)** The model must have an objective function.
- **(c)** The model must have structural constraints.
- **(d)** The model must have non-negativity constraint.

Let us consider a product mix problem and see the applicability of the above properties.

Example .1. A company manufactures two products *X* and *Y*, which require, the following resources. The resources are the capacities machine M_1, M_2, and M_3. The available capacities are 50,25,and 15 hours respectively in the planning period. Product *X* requires 1 hour of machine M_2 and 1 hour of machine M_3. Product *Y* requires 2 hours of machine M_1, 2 hours of machine M_2 and 1 hour of machine M_3. The profit contribution of products *X* and *Y* are Rs.5/- and Rs.4/- respectively.

The contents of the statement of the problem can be summarized as follows:

Machines	*Products*		*Availability in hours*
	X	*Y*	
M_1	0	2	50
M_2	1	2	25
M_3	1	1	15
Profit in Rs. Per unit	5	4	

In the above problem, Products *X* and *Y* are competing candidates or variables. Machine capacities are available resources. Profit contribution of products *X* and *Y* are given.

Now let us formulate the model.

Let the company manufactures *x* units of *X* and y units of *Y*. As the profit contributions of *X* and *Y* are Rs.5/- and Rs. 4/- respectively. The objective of the problem is to maximize the profit *Z*, hence objective function is:

Maximize $Z = 5x + 4y$ → OBJECTIVE FUNCTION.

This should be done so that the utilization of machine hours by products x and y should not exceed the available capacity. This can be shown as follows:

For Machine M_1 $0x + 2y \leq 50$

For Machine M_2 $1x + 2y \leq 25$ and → LINEAR STRUCTURAL CONSTRAINTS.

For machine M_3 $1x + 1y \leq 15$

But the company can stop production of *x* and *y* or can manufacture any amount of *x* and *y*. It cannot manufacture negative quantities of *x* and *y*. Hence we have write,

Both *x* and *y* are $\geq$ 0 . → NON-NEGATIVITY CONSTRAINT.

As the problem has got objective function, structural constraints, and non-negativity constraints and there exist a linear relationship between the variables and the constraints in the form of inequalities, the problem satisfies the properties of the Linear Programming Problem.

Basic Assumptions

The following are some important assumptions made in formulating a linear programming model:

1. It is assumed that the decision maker here is *completely certain* (*i.e.*, deterministic conditions) regarding all aspects of the situation, *i.e.*, availability of resources, profit contribution of the products, technology, courses of action and their consequences etc.
2. It is assumed that the relationship between variables in the problem and the resources available *i.e.*, constraints of the problem exhibits *linearity*. Here the term linearity implies proportionality and additivity. This assumption is very useful as it simplifies modeling of the problem.
3. We assume here *fixed technology*. Fixed technology refers to the fact that the production requirements are fixed during the planning period and will not change in the period.
4. It is assumed that the *profit contribution of a product remains constant*, irrespective of levelof production and sales.
5. It is assumed that the decision variables are *continuous*. It means that the companies manufacture products in fractional units. For example, company manufacture 2.5 vehicles,
   3.2 barrels of oil etc. This is referred too as the assumption of *divisibility*.
6. It is assumed that *only one decision* is required for the planning period. This condition shows that the linear programming model is a static model, which implies that the linear programming problem is a *single stage decision problem*. (Note: Dynamic Programming problem is a multistage decision problem).
7. All variables are restricted to *nonnegative values* (*i.e.*, their numerical value will be ≥ 0).

Terms Used in Linear Programming Problem

Linear programming is a method of obtaining an optimal solution or programme (say, product mix in a production problem), when we have limited resources and a good number of competing candidates to consume the limited resources in certain proportion. The term linear implies the condition of proportionality and additivity. The programme is referred as a course of action covering a specified period of time, say planning period. The manager has to find out the best course of action in the interest of the organization. This best course of action is termed as optimal course of action or optimal solution to the problem. A programme is optimal, when it maximizes or minimizes some measure or criterion ofeffectiveness, such as profit, sales or costs.

The term programming refers to a systematic procedure by which a particular program or plan ofaction is designed. Programming consists of a series of instructions and computational rules for solving a problem that can be worked out manually or can fed into the computer. In solving linear programming problem, we use a systematic method known as simplex method developed by American mathematician George B. Dantzig in the year 1947.

The candidates or activity here refers to number of products or any such items, which need the utilization of available resources in a certain required proportion. The available resources may be of any nature, such as money, area of land, machine hours, and man-hours or materials. But they are limited in availability and which are desired by the activities / products for consumption.

General Linear Programming Problem

A general mathematical way of representing a Linear Programming Problem (L.P.P.) is as given below:

$Z = c_1x_1 + c_2 x_2 + \dots c_n x_n$ subjects to the conditions, $\longrightarrow$ **OBJECTIVE FUNCTION**

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \dots\dots\dots\dots + a_{2j} x_j + \dots\dots + a_{2n}x_n \;(\geq, =, \leq)\; b_2$ **Structural**

....

....

and all x_j are = 0 $\longrightarrow$ **NON NEGETIVITY CONSTRINT.**

Where $j = 1, 2, 3, \dots n$

Where all c_j s, b_i s and a_{ij} s are constants and x_j s are decision variables.

To show the relationship between left hand side and right hand side the symbols □ , = , □ are used. Any one of the signs may appear in real problems. Generally □ sign is used for maximization problems and sign is used for minimization problems and in some problems, which are known as mixed problems we may have all the three signs. The word optimize in the above model indicates either maximise or minimize. The linear function, which is to be optimized, is the objective function. The inequality conditions shown are constraints of the problem. Finally all x_i s should be positive, hence the non-negativity function.

The steps for formulating the linear programming are:

1. Identify the unknown decision variables to be determined and assign symbols to them.
2. Identify all the restrictions or constraints in the problem and express them as linearequations or inequalities of decision variables.
3. Identify the objective or aim and represent it also as a linear function of decision variables.

Construct linear programming model for the following *problems*:

MAXIMIZATION MODELS

Example 2. A retail store stocks two types of shirts *A* and *B*. These are packed in attractive cardboard boxes. During a week the store can sell a maximum of 400 shirts of type *A* and a maximum of 300 shirts of type *B*. The storage capacity, however, is limited to a maximum of 600 of both types combined. Type *A* shirt fetches a profit of Rs. 2/- per unit and type *B* a profit of Rs. 5/- per unit. How many of each type the store should stock per week to maximize the total profit? Formulate a mathematical model of the problem**.**

Solution: Here shirts *A* and *B* are problem variables. Let the store stock '*a*' units of *A* and '*b*' units of
B. As the profit contribution of *A* and *B* are Rs.2/- and Rs.5/- respectively, objective function is: Maximize $Z = 2a + 5b$ subjected to condition (s.t.)

Structural constraints are, stores can sell 400 units of shirt *A* and 300 units of shirt *B* and the storage capacity of both put together is 600 units. Hence the structural constraints are:

$1a + 0b \geq 400$ and $0a + 1b$ □ 300 for sales capacity and $1a + 1b$ □ 600 for storage capacity. And non-negativity constraint is both *a* and *b* are □ 0. Hence the model is:

Maximize $Z = 2a + 5b$ s.t.

$1a + 0b$ □ 400

$0a + 1b$ □ 300

$1a + 1b$ □ 600
and Both *a* and *b* are □ 0.

Problem 3. A ship has three cargo holds, forward, aft and center. The capacity limits are: Forward 2000 tons, 100,000 cubic meters
Center 3000 tons, 135,000 cubic meters
Aft 1500 tons, 30,000 cubic meters.

The following cargoes are offered, the ship owners may accept all or any part of each commodity:

Commodity	Amount in tons.	Volume/ton in cubic meters	Profit per ton in Rs.
A	6000	60	60
B	4000	50	80
C	2000	25	50

In order to preserve the trim of the ship the weight in each hold must be proportional to the capacity in tons. How should the cargo be distributed so as to maximize profit? Formulate this as linear programming problem.

Solution: Problem variables are commodities, *A*, *B*, and *C*. Let the shipping company ship '*a*' units of
A and '*b'* units of *B* and '*c'* units of *C*. Then Objective function is:

Maximize $Z = 60a + 80b + 50c$ s.t.

Constraints are:

Weight constraint: $6000a + 4000b + 2000c$ □ 6,500 (= 2000+3000+1500)

The tonnage of commodity is 6000 and each ton occupies 60 cubic meters, hence there are 100 cubic meters capacity is available.

Similarly, availability of commodities *B* and *C*, which are having 80 cubic meter capacities each.

Hence capacity inequality will be:

$100a + 80b + 80c$ □ 2,65,000 (= 100,000+135,000+30,000). Hence the l.p.p. Model is:

Maximise $Z = 60a+80b+50c$ s.t. $100a = 6000/60 = 100$

$6000a + 4000b + 2000c$ □ 6,500 $80b = 4000/50 = 80$

$100a+80b+80c$ □ 2,65,000 and $80c = 2000/25 = 80$ etc.

a,b,c all □ 0

MINIMIZATION MODELS

Problem 4. A patient consult a doctor to check up his ill health. Doctor examines him and advises him that he is having deficiency of two vitamins, vitamin *A* and vitamin *D*. Doctor advises him to consume vitamin *A* and *D* regularly for a period of time so that he can regain his health. Doctor prescribes tonic *X* and tonic *Y*, which are having vitamin *A*, and *D* in certain proportion. Also advises the patient to consume **at least** 40 units of vitamin *A* and 50 units of vitamin Daily. The cost of tonics *X* and *Y* and the proportion of vitamin *A* and *D* that present in *X* and *Y* are given in the table below. Formulate l.p.p. to minimize the cost of tonics.

Vitamins	Tonics		Daily requirement in units.
	X	*Y*	
A	2	4	40
D	3	2	50
Cost in Rs. per unit.	5	3	

Solution: Let patient purchase x units of *X* and y units of *Y*.

Objective function: Minimize $Z = 5x + 3y$

Inequality for vitamin *A* is $2x + 4y$ □ 40 (Here **at least** word indicates that the patient canconsume more than 40 units but not less than 40 units of vitamin A daily).

Similarly the inequality for vitamin D is $3x + 2y$ □ 50.

For non–negativity constraint the patient cannot consume negative units. Hence both x and y must be □ 0.

Now the l.p.p. model for the problem is:

Minimize $Z = 5x + 3y$

s.t. $2x + 4y$ □ 40

$3x + 2y$ □ 50 and

Both x and y are □ 0.

Problem 5. A machine tool company conducts a job-training programme at a ratio of one for every ten trainees. The training programme lasts for one month. From past experience it has been found that out of 10 trainees hired, only seven complete the programme successfully. (The unsuccessful trainees are released). Trained machinists are also needed for machining. The company's requirement for the next three months is as follows:

January: 100 machinists, February: 150 machinists and March: 200 machinists.

In addition, the company requires 250 trained machinists by April. There are 130 trained machinists available at the beginning of the year. Pay roll cost per month is:

Each trainee Rs. 400/- per month.

Each trained machinist (machining or teaching): Rs. 700/- p.m. Each trained machinist who is idle: Rs.500/- p.m.

(Labour union forbids ousting trained machinists). Build a l.p.p. for produce the minimum cost hiring and training schedule and meet the company's requirement.

Solution: There are three options for trained machinists as per the data given. (*i*) A trained machinist can work on machine, (*ii*) he can teach or (*iii*) he can remain idle. It is given that the number of trained machinists available for machining is fixed. Hence the unknown decision variables are the number of machinists goes for teaching and those who remain idle for each month. Let,

'*a*' be the trained machinists teaching in the month of January. '*b*' be the trained machinists idle in the month of January.

'*c'* be the trained machinists for teaching in the month of February. '*d'* be the trained machinists remain idle in February.

'*e'* be the trained machinists for teaching in March.

'*f'* be the trained machinists remain idle in the month of March.

The constraints can be formulated by the rule that the number of machinists used for (machining + teaching + idle) = Number of trained machinists available at the beginning of the month.

For January $100 + 1a + 1b \;\square\; 130$

For February, $150 + 1c + 1d = 130 + 7a$ (Here $7a$ indicates that the number of machinist trained is $10 \times a = 10a$. But only 7 of them are successfully completed the training *i.e*. $7a$).

For the month of March, $200 + 1e + 1f \;\square\; 130 + 7a + 7c$

The requirement of trained machinists in the month of April is 250, the constraints for this will be $130 + 7a + 7c + 7e \;\square\; 250$ and the objective function is

Minimize $Z = 400\,(10a + 10c + 10e) + 700\,(1a + 1c + 1e) + 400\,(1b + 1d + 1f)$ and the non-negativity constraint is a, b, c, d, e, f all $\square$ 0. The required model is:

Minimize $Z = 400\,(10a + 10c + 10e) + 700\,(1a + 1c + 1e) + 400\,(1b + 1d + 1f)$ s.t. $100 + 1a + 1b \;\square\; 130$

$150 + 1c + 1d \;\square\; 130 + 7a$

$200 + 1e + 1f \;\square\; 130 + 7a + 7c$

$130 + 7a + 7c + 7e \;\square\; 250$ and

a, b, c, d, e, f all $\square$ 0.

METHODS FOR THE SOLUTION OF A LINEAR PROGRAMMING PROBLEM

Linear Programming, is a method of solving the type of problem in which two or more **candidates** or **activities** are competing to utilize the available limited resources, with a view to **optimize** the **objective function** of the problem. The objective may be to maximize the **returns** or to minimize the **costs**. The various methods available to solve the problem are:

1. The Graphical Method when we have two decision variables in the problem. (To deal with more decision variables by graphical method will become complicated, because we have to deal with planes instead of straight lines. Hence in graphical method let us limit ourselves to two variable problems.
2. The Systematic Trial and Error method, where we go on giving various values to variables until we get optimal solution. This method takes too much of time and laborious, hence thismethod is not discussed here.
3. The Vector method. In this method each decision variable is considered as a vector and principles of vector algebra is used to get the optimal solution. This method is also time consuming, hence it is not discussed here.
4. The Simplex method. When the problem is having more than two decision variables, simplex method is the most powerful method to solve the problem. It has a systematic programme, which can be used to solve the problem.

One problem with two variables is solved by using both graphical and simplex method, so as to enable the reader to understand the relationship between the two.

Graphical Method

In graphical method, the inequalities (structural constraints) are considered to be equations. This is because; one cannot draw a graph for inequality. Only two variable problems are considered, because we can draw straight lines in two-dimensional plane (*X*- axis and *Y*-axis). More over as we have non- negativity constraint in the problem that is all the decision variables must have positive values always the solution to the problem lies in first quadrant of the graph. Some times the value of variables may fall in quadrants other than the first quadrant. In such cases, the line joining the values of the variables must be extended in to the first quadrant. The procedure of the method will be explained in detail while solving a numerical problem. The characteristics of Graphical method are:

(i) Generally the method is used to solve the problem, when it involves two decision variables.

(ii) For three or more decision variables, the graph deals with planes and requires high imaginationto identify the solution area.

(iii) Always, the solution to the problem lies in first quadrant.

(iv) This method provides a basis for understanding the other methods of solution.

Problem 6. A company manufactures two products, *X* and *Y* by using three machines *A*, *B*, and *C*. Machine *A* has 4 hours of capacity available during the coming week. Similarly, the available capacity of machines *B* and *C* during the coming week is 24 hours and 35 hours respectively. One unit of

product *X* requires one hour of Machine *A*, 3 hours of machine *B* and 10 hours of machine *C*. Similarly one unit of product *Y* requires 1 hour, 8 hour and 7 hours of machine *A*, *B* and *C* respectively. When one unit of *X* is sold in the market, it yields a profit of Rs. 5/-

per product and that of *Y* is Rs. 7/- per unit. Solve the problem by using graphical method to find the optimal product mix.

Solution: The details given in the problem is given in the table below:

Machines	*Products (Time required in hours).*		*Available capacity in hours.*
	X	*Y*	
A	1	1	4
B	3	8	24
C	10	7	35
Profit per unit in Rs.	5	7	

Let the company manufactures x units of X and y units of Y, and then the L.P. model is: Maximise $Z = 5x + 7y$ s.t.

$1x + 1y = 4$, $3x + 8y = 24$, $10x + 7y = 35$ and Both x and y are = 0.

As we cannot draw graph for inequalities, let us consider them as equations. Maximise $Z = 5x + 7y$ s.t.

$1x + 1y = 4$, $3x + 8y = 24$

$10x + 7y = 35$ and both x and y are □ 0

Let us take machine *A*. and find the boundary conditions. If $x = 0$, machine *A* can manufacture 4/1 = 4 units of y.

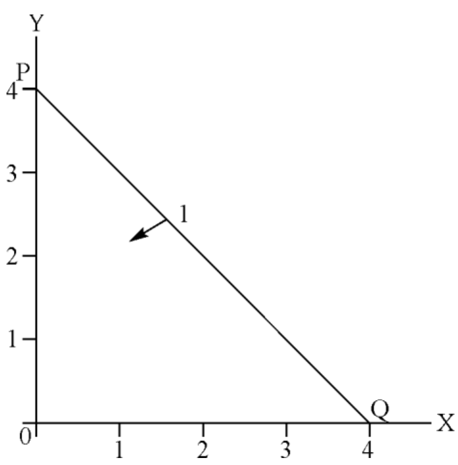


Figure 1 Graph for machine A

Similarly, if $y = 0$, machine *A* can manufacture 4/1 = 4 units of x. For other machines: Machine *B* When $x = 0$, $y = 24/8 = 3$ and when $y = 0$ $x = 24/3 = 8$

Machine *C* When $x = 0$, $y = 35/10 = 3.5$ and when $y = 0$, $x = 35 / 7 = 5$.

These values we can plot on a graph, taking product *X* on x-axis and product *Y* on y- axis. First let us draw the graph for machine *A*. In figure 2. 1 we get line 1 which represents $1x + 1y$

= 4. The point *P* on *Y* axis shows that the company can manufacture 4 units of *Y* only when does not want to manufacture *X*. Similarly the point *Q* on *X* axis shows that the company can manufacture 4 units of *X*, when does not want to manufacture *Y*. In fact triangle *POQ* is the capacity of machine *A* and the line *PQ* is the boundary line for capacity of machine *A*.

Similarly figure 2.2 show the Capacity line RS for machine *B*. and the triangle *ROS* shows the capacity of machine *B i.e.*, the machine *B* can manufacture 3 units of product *Y* alone or 8 units of product *X* alone.

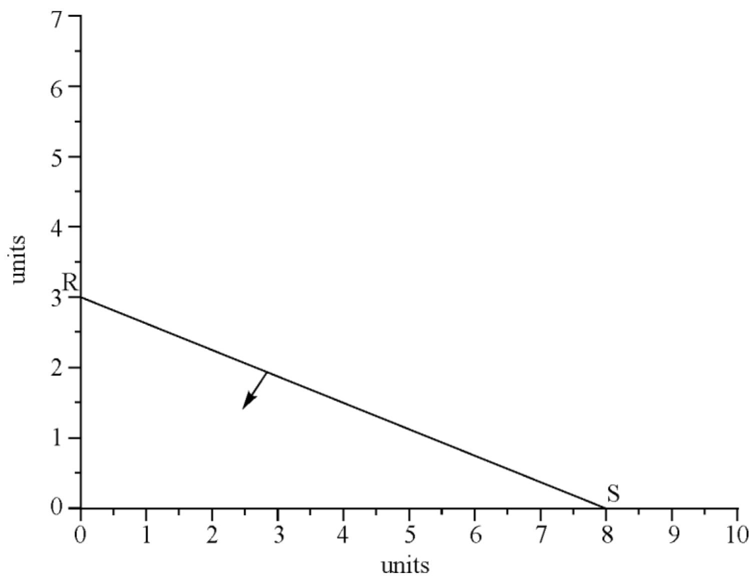


Figure 2. Graph for machine B

The graph 2 shows that the machine *C* has a capacity to manufacture 5 units of *Y* alone or 3.5 units of *X* alone. Line *TU* is the boundary line and the triangle *TOU* is the capacity of machine *C*.

The graph is the combined graph for machine *A* and machine *B*. Lines *PQ* and RS intersect at *M*. The area covered by both the lines indicates the products (*X* and *Y*) that can be manufactured by using both machines. This area is the feasible area, which satisfies the conditions of inequalities of machine *A* and machine *B*. As *X* and *Y* are processed on *A* and *B* the number of units that can be manufactured will vary and the there will be some idle capacities on both machines.

Method 1. Here we find the co-ordinates of corners of the closed polygon *ROUVW* and substitute the values in the objective function. In maximisaton problem, we select the co-ordinates giving maximum value. And in minimisaton problem, we select the co-ordinates, which gives minimum value.

In the problem the co-ordinates of the corners are:

R = (0, 3.5), O = (0,0), U = (3.5,0), V = (2.5, 1.5) and W = (1.6,2.4). Substituting these values in objective function:

$$Z_{(0,3.5)} = 5 \times 0 + 7 \times 3.5 = \text{Rs. } 24.50, \text{ at point } R$$

$$Z_{(0,0)} = 5 \times 0 + 7 \times 0 = \text{Rs. } 00.00, \text{ at point } O$$

$$Z_{(3.5,0)} = 5 \times 3.5 + 7 \times 0 = \text{Rs. } 17.5 \text{ at point } U$$

$$Z_{(2.5,\ 1.5)} = 5 \times 2.5 + 7 \times 1.5 = \text{Rs. } 23.00 \text{ at point } V$$

$$Z_{(1.6,\ 2.4)} = 5 \times 1.6 + 7 \times 2.4 = \text{Rs. } 24.80 \text{ at point } W$$

Hence the optimal solution for the problem is company has to manufacture 1.6 units of product
X and 2.4 units of product *Y*, so that it can earn a maximum profit of Rs. 24.80 in the planning period.

Method 2. Iso profit Line Method: Isoprofit line, a line on the graph drawn as per the objective function, assuming certain profit. On this line any point showing the values of x and y will yield sameprofit. For example in the given problem, the objective function is Maximise $Z = 5x + 7y$. If we assume a profit of Rs. 35/-, to get Rs. 35, the company has to manufacture either 7 units of *X* or 5 units of *Y*.

Hence, we draw line *ZZ* (preferably dotted line) for $5x + 7y = 35$. Then draw parallel line to this line *ZZ* at origin. The line at origin indicates zero rupees profit. No company will be willing to earn zero rupees profit. Hence slowly move this line away from origin. Each movement shows a certain profit, which is greater than Rs.0.00. While moving it touches corners of the polygon showing certain higher profit. Finally, it touches the farthermost corner covering all the area of the closed polygon. This point where the line passes (farthermost point) is the **OPTIMAL SOLUTION** of the problem. In the figure 2.6. the line *ZZ* passing through point *W* covers the entire area of the polygon, hence it is the point that yields highest profit. Now point *W* has co-ordinates (1.6, 2.4)**. Now Optimal profit $Z = 5 \times 1.6 + 7 \times 2.4$ = Rs. 24.80.**

Problem 7. Solve graphically the given linear programming problem. (Minimization Problem).

Minimize $Z = 3a + 5b$ S.T

$-3a + 4b$ □ 12

$2a - 1b$ □ -2

$2a + 3b$ □ 12

$1a + 0b$ □ 4

$0a + 1b$ □ 2

Points to be Noted:

- **(i)** **In inequality $-3a + 4b \le 12$, product/the candidate/activity requires –3 units of the resource. It does not give any meaning (or by manufacturing the product A the manufacturer can save 3 units of resource No.1 or one has to consume –3 units of A. (All these do not give any meaning as far as the practical problems or real world problems are concerned).**
- **(ii)** **In the second inequality, on the right hand side we have –2. This means that –2 units of resource is available. It is absolutely wrong. Hence in solving a l.p.p. problem, one must see that the right hand side we must have always a positive integer. Hence the inequality is to be multiplied by –1 so that the inequality sign also changes. In the present case it becomes: $-2a + 1b \le 2$.**

And both *a* and *b* are □ 0.

Solution: Now the problem can be written as:

Minimize $Z = 3a + 5b$ S.T.

When converted into equations they can be written as Min. $Z = 3a + 5b$ *S.T.*

$-3a + 4b$ □ 12 $\quad\quad\quad\quad$ $-3a + 4b = 12$

$-2a + 1b$ □ 2 $\quad\quad\quad\quad$ $-2a + 1b = 2$

$2a - 3b$ □ 12 $\quad\quad\quad\quad$ $2a - 3b = 12$

$1a + 0b$ □ 4 $\quad\quad\quad\quad$ $1a + 0b = 4$

$0a + 1b$ □ 2 and both a and b are $\geq = 0.0a + 1b \geq 2$ and both a and b are $\geq$ 0.

The lines for inequalities $-3a + 4b \leq 12$ and $-2a + 1b \leq 2$ starts from quadrant 2 and they are to be extended into quadrant 1. Figure 2.7 shows the graph, with Isocost line.

Iso cost line is a line, the points on the line gives the same cost in Rupees. We write Isocost line at a far off place, away from the origin by assuming very high cost in objective function. Then we move line parallel towards the origin (in search of least cost) until it passes through a single corner of the closed polygon, which is nearer to the origin, (Unique Solution), or passes through more than one point, which are nearer to the origin (more than one solution) or coincides with a line nearer to the origin and the side of the polygon (innumerable solution). The solution for the problem is the point P (3,2,) and the Minimum cost is Rs. $3 \times 3 + 2 \times 5$ = Rs. 19/-

Problem 8. A company manufactures two products X and Y. The profit contribution of X and Y are Rs.3/- and Rs. 4/- respectively. The products X and Y require the services of four facilities. The capacities of the four facilities A, B, C, and D are limited and the available capacities in hours are 200 Hrs, 150 Hrs, and 100 Hrs. and 80 hours respectively. Product X requires 5, 3, 5 and 8 hours of facilities A, B, C and D respectively. Similarly the requirement of product Y is 4, 5, 5, and 4 hours respectively on A, B, C and D. Find the optimal product mix to maximise the profit.

Solution: Enter the given data in the table below:

	product s		
Machines	*X*	*Y*	*Availability in hours.*
	(Time in hours)		
A	5	4	200
B	3	5	150
C	5	4	100
D	8	4	80
Profit in Rs. Per unit:	3	4	

The inequalities and equations for the above data will be as follows. Let the company manufactures
x units of X and y units of Y.

Maximise Z $3x + 4y$ S.T. $\quad\quad\quad\quad$ Maximise $Z = 3x + 4y$ S.T.

$5x + 4y$ □ 200 $\quad\quad\quad\quad$ $5x + 4y = 200$

$3x + 5y$ □ 150	$3x + 5y = 150$
$5x + 4y$ □ 100	$5x + 4y = 100$
$8x + 4y$ □ 80	$8x + 4y = 80$
And both x and y are □ 0	And both x and y are □ 0

In the graph the line representing the equation $8x + 4y$ is out side the feasible area and hence it is a redundant equation. It does not affect the solution. The Isoprofit line passes through corner T of the polygon and is the point of maximum profit. Therefore $Z_T = Z_{(32,10)} = 3 \times 32 + 4 \times 10 =$ Rs. 136/.

SIMPLEX METHOD

There are many methods to solve the Linear Programming Problem, such as Graphical Method, Trial and Error method, Vector method and Simplex Method. Though we use graphical method for solution when we have two problem variables, the other method can be used when there are more than two decision variables in the problem. Among all the methods, SIMPLEX METHOD is most powerful method. It deals with iterative process, which consists of first designing a Basic Feasible Solution or a Programme and proceed towards the OPTIMAL SOLUTION and testing each feasible solution for Optimality to know whether the solution on hand is optimal or not. If not an optimal solution, redesign the programme, and test for optimality until the test confirms OPTIMALITY. Hence we can say that the Simplex Method depends on two concepts known as Feasibility and optimality.

The simplex method is based on the property that the optimal solution to a linear programming problem, if it exists, can always be found in one of the basic feasible solution. The simplex method is quite simple and mechanical in nature. The iterative steps of the simplex method are repeated until a finite optimal solution, if exists, is found. If no optimal solution, the method indicates that no finite solution exists.

MAXIMISATION CASE

Problem 9: A factory manufactures two products *A* and *B* on three machines *X*, *Y*, and *Z*. Product A requires 10 hours of machine *X* and 5 hours of machine *Y* a one our of machine *Z*. The requirement of product *B* is 6 hours, 10 hours and 2 hours of machine *X*, *Y* and *Z* respectively. The profit contribution of products *A* and *B* are Rs. 23/– per unit and Rs. 32 /– per unit respectively. In the coming planning period the available capacity of machines *X*, *Y* and *Z* are 2500 hours, 2000 hours and 500 hours respectively. Find the optimal product mix for maximizing the profit.

Solution:

The given data is:

Machines	*Products*		*Capacity in hours*
	A *Hrs.*	*B* *Hrs.*	
X	10	6	2500
Y	5	10	2000

Z	1	2	500
Profit/unit Rs.	23	32	—

Let the company manufactures a units of A and b units of B. Then the inequalities of the constraints (machine capacities) are:

Maximise $Z = 23\,a + 32\,b$ S.T. → OBJECTIVE

FUNCTION $10\,a + 6\,b$ □ 2500

$5\,a + 10\,b$ □ 2000 → STRUCTURAL CONSTRAINTS.

$1\,a + 2\,b$ □ 500

And both a and b are □ 0. → NON-NEGATIVITY CONSTRAINT.

Now the above inequalities are to be converted into equations.

Take machine X: One unit of product A requires 10 hours of machine X and one unit of product B require 6 units. But company is manufacturing a units of A and b units of B, hence both put together must be **less than or equal to** 2,500 hours. Suppose a = 10 and b = 10 then the total consumption is $10 \times 10 + 6 \times 10 = 160$ hours. That is out of 2,500 hours, 160 hours are consumed, and 2,340 hours

are still remaining idle. So if we want to convert it into an equation then 100 + 60 + 2,340 = 2,500. As we do not know the exact values of decision variables a and b how much to add to convert the inequality into an equation. For this we represent the idle capacity by means of a **SLACK VARIABLE** represented by **S**. Slack variable for first inequality is S_1, that of second one is S_2 and that of 'n'th inequality is S_n.

Regarding the objective function, if we sell one unit of A it will fetch the company Rs. 23/– per

unit and that of B is Rs. 32/– per unit. If company does not manufacture A or B, all resources remain idle. Hence the profit will be Zero rupees. This clearly shows that the profit contribution of each hour of idle resource is zero. In Linear Programming language, we can say that the company has capacity of manufacturing 2,500 units of S_1, *i.e.*, S_1 is an **imaginary product**, which require one hour of machine X alone. Similarly, S_2 is an **imaginary product** requires one hour of machine Y alone and S_3 is an imaginary product, which requires one hour of machine Z alone. In simplex language S_1, S_2 and S_3 are idle resources. The profit earned by keeping all the machines idle is Rs.0/–. Hence the profit contributions of S_1, S_2 and S_3 are Rs.0/– per unit. By using this concept, the inequalities are converted into equationsas shown below:

Maximise $Z = 23\,a + 32\,b + 0S_1 + 0S_2 + 0S_3$ S.T.

$$10\,a + 6\,b + 1S_1 = 2500$$

$$5\,a + 10\,b + 1S_2 = 2000$$

$1\,a + 2\,b + 1S_3 = 500$ and a, b, S_1, S_2 and S_3 all □ 0.

In Simplex version, all variables must be available in all equations. Hence the Simplex format ofthe model is:

Maximise $Z = 23\ a + 32\ b + 0S_1 + 0S_2 + 0S_3$ S.T. $10\ a + 6\ b + 1S_1 + 0S_2 + 0S_3 = 2500$

$5\ a + 6\ b + 0S_1 + 1S_2 + 0S_3 = 2000$

$1\ a + 2\ b + 0S_1 + 0S_2 + 1S_3 = 500$ and a, b, S_1, S_2 and S_3 all ≥ 0.

The above data is entered in a table known as **simplex table (or tableau)**. There are many versions of table but in this book only one type is used.

In Graphical method, while finding the profit by Isoprofit line, we use to draw Isoprofit line at origin and use to move that line to reach the far off point from the origin. This is because starting from zero rupees profit; we want to move towards the maximum profit. Here also, first we start with zero rupees profit, *i.e.*, considering the slack variables as the basis variables (problem variables) in the initial programme and then improve the programme step by step until we get the optimal profit. Let us start the first programme or initial programme by rewriting the entries as shown in the above simplex table.

MINIMISATION CASE

Above we have discussed how to solve maximisation problem and the mechanism or simplex method and interpretation of various elements of rows and columns. Now let us see how to solve a minimization problem and see the mechanism of the simplex method in solving and then let us deal with some typical examples so as to make the reader confident to be confident enough to solve problem individually.

Comparison between maximisaton case and minimisation case

S.No.	Maximisation case	Minimisation case
	Similarities:	
1.	It has an objective function.	This too has an objective function.
2.	It has structural constraints.	This too has structural constraints.
3.	The relationship between variables and constraints is linear.	Here too the relationship between and variables constraints is linear.
4.	It has non-negativity constraint.	This too has non-negativity constraints.
5.	The coefficients of variables may be positive or negative or zero.	The coefficient of variables may be positive, Negative or zero.
6.	For selecting out going variable (key row) lowest replacement ratio is selected.	For selecting out going variable (key row) lowest replacement ratio is selected.
	Differences:	
1.	The objective function is of maximisation type.	The objective function is of minimisation type.
2.	The inequalities are of $\leq$ type.	The inequalities are of $\geq$ type.
3.	To convert inequalities into equations, slack variables are added.	To convert inequalities into equations, surplus Variables are subtracted and artificial surplus variables are added.
4.	While selecting incoming variable, highest positive Opportunity cost is	While slecting, incoming variable, lowest element in the net evaluation row

	selected from net evaluation Row.	is selected (highest number with negative sign).
5.	When the elements of net evaluation row are either Negative or zeros, the solution is optimal	When the element of net evaluation row are either positive or zeros the solution is optimal.

It is most advantageous to introduce minimisation problem by dealing with a well-known problem, known as **diet problem.**

Problem 2: In this problem, a patient visits the doctor to get treatment for ill health. The doctor examines the patient and advises him to consume **at leas**t 40 units of vitamin *A* and 50 units of vitamin *B* daily for a specified time period. He also advises the patient that to get vitamin *A* and vitamin *B* he has to drink tonic *X* and tonic *Y* that have both vitamin *A* and vitamin *B* in a proportion. One unit of tonic *X* consists 2 units of vitamin *A* and 3 units of vitamin *B* and one unit of tonic *Y* consists of 4 units of vitamin *A* and 2 units of vitamin *B*. These tonics are available in medical shops at a cost of Rs.3.00 and Rs.2.50 per unit of *X* and *Y* respectively. Now the problem of patient is how much of *X* and how much of *Y* is to be purchased from the shop to **minimise** the total cost and at the same time he can get **required amounts** of vitamins *A* and *B*.

First we shall enter all the data in the form of a table.

Vitamin	Tonic		Requirement
	X	*Y*	
A	2	4	40
B	3	2	50
Cost in Rs.	3	2.50	

Let the patient purchase 'x' units of *X* and 'y' units of *Y* then the inequalities are **(Note: the condition given in the problem is AT LEAST hence the inequalities are of ≥ type)**

Inequalities:

For vitamin A: Minimize $Z = 3x + 2.5$

y S.T $2x + 4y \geq 40$

$3x + 2y \geq 50$

And both x and y are ≥ 0.

In the above inequalities, say $2x + 4y \geq 40$, if we give values to x and y such that the sum is greater than or equal to 40, for example, $x = 10$ and $y = 10$ then $2x + 4y = 60$ which is > 40. To make it equal to 40 we have to subtract 20, so that 20 + 40 – 20 = 40. When we know the values, we can do this. But as we do not know the values of x and y we have to subtract a **SURPLUS VARIABLE**, generally represented by 'p', 'q', 'r'……. etc. If we do this then the inequality $2x + 4y \geq 40$ will be $2x + 4y - 1p = 40$.

Now if we allocate value **zero** to x and y then $0 + 0 - 1p = 40$ or $p = -40$. Which is against to the rules of l.p.p. as every l.p.problem the values of variables must be ≥ 0. Hence in minimization problem, we introduce one more **Surplus variable**, known as

ARTIFICIAL SURPLUS VARIABLE generally represented by A_1, A_2, A_3 … etc. Now by introducing artificial surplus variable, we can write $2x + 4y = 40$ as $\mathbf{2x + 4y - 1p + 1A_1 = 40}$.

If values of x, y, and p are equal to zero, then $\mathbf{1A_1 = 40}$. The artificial surplus variable has the value 40, a positive integer. Hence we start our initial programme with the artificial variables, A_1, A_2, A_3 etc. and go on replacing them by x, y, z etc. that is decision variables.

Coming to the cost coefficients of surplus and artificial surplus variables, for example, p **is very similar to vitamin *A* and one unit of *p* consists of only one unit of vitamin *A***. It will come as **give away** product when we purchase vitamin *A*. That is the cost coefficient of 'p' is zero (it is very much similar to slack variable in maximization problem). But the artificial surplus variable has to be purchased by paying a very high price for it. In character it is very much similar to surplus variable 'p' because one unit of A_1 consists of one unit of vitamin A. The cost coefficient of A_1 is represented by a very high value represented by M (which means one unit of A_1 cost Millions or Rupees). As we are introducing **CAPITAL 'M', THIS METHOD IS KNOWN AS BIG 'M' METHOD**.

By using the above concept, let us write the equations of the inequalities of the problem. Minimise $Z = 3x + 2.5y + 0p + 0q + MA_1 + MA_2$ S.T. → Objective Function.

$2x + 4y - 1p + 1A_1 = 40$

$3x + 2y - 1q + 1A_2 = 50$ → Structural Constraints.

And x, y, p, q, A_1, A_2 all ≥ 0 → Non negativity Constraint.

Simplex format of the above is:

Minimise $Z = 3x + 2.5y + 0p + 0q + MA_1 + MA_2$ S.T. $2x + 4y - 1p + 0q + 1A_1 + 0A_2 = 40$

$3x + 2y + 0p - 1q + 0A_1 + 1A_2 = 50$

And x, y, p, q, A_1, A_2 all = 0.

Let us enter the data in the Initial table of Simplex method.

Table: 1.

$x = 0$, $y = 0$, $p = 0$, $q = 0$ $A_1 = 40$, $Z = \text{Rs. } 40M + 50M = 90M$

Programme variable	*Cost per unit in Rs.*	*Cost requirement*	*Cj* 3 *x*	2.5 *y*	0 *p*	0 *q*	*M* A_1	*M* A_2	*Replacement ratio*
A_1	*M*	40	2	**4**	– 1	0	1	0	40/4 = 10
A_2	*M*	50	3	2	0	– 1	0	1	50/2 = 25
Z_j			5 *M*	6 *M*	– *M*	– *M*	*M*	*M*	

Net evaluation	$C_j - Z_j$		$3 - 5M$	$2.5 - 6M$	M	M	0	0		

Note: As the variables A_1 and A_2 are basis variables, their Net evaluation is zero.

Now take 6M and 5M, 6 M is greater and if we subtract 2.5 from that it is negligible. Hence –6m will be the lowest element. The physical interpretation is if patient purchases *Y* now, his cost will be reduced by an amount 6M. In other words, if the patient does not purchase the *Y* at this point, his penalty is 6M, *i.e.*, **the opportunity cost is 6M.** As the **non-basis variable *Y* has highest opportunity cost** (highest element with negative sign), *Y* is the **incoming variable**. Hence, the column under *Y* is **key column**. To find the out going variable, divide requirement column element by **key column** element and find the replacement ratio. Select the lowest ratio, *i.e*., here it is 10, falls in first row, hence A_1 is the out going variable.

To transfer key row, divide all the elements of key row by key number (= 4).

40/4 = 10, 2/4 = 0.5, –1/4 = – 0.25, 0/25 = 0, 1/25 = 0.25, 0/4 = 0.

To transfer non-key row elements:

New row element = old row element – corresponding Key row element × (Key column number/key number).

50 – 40 × 2/4 = 30

3 – 2 × 0.5 = 2

2 – 4 × 0.5 = 0

0 – (–1) × 0.5 = 0.5 .

–1 – 0 × 0.5 = – 1

0 – 1 × 0.5 = – 0.5

1 – 0 × 0.5 = 1

Note:

(i) The elements under A_1 and A_2 *i.e*., artificial variable column are negative versions of elements under artificial variable column.

(ii) The net evaluation row elements of basis variables are always zero. While writing the second table do not change the positions of the rows).

Let us now enter the new elements of changed rows in the second simplex table.

Table: 2.

$x = 0$, $y = 10$, $p = 0$, $q = 0$, $A_1 = 0$, $A_2 = 30$ and Z = Rs. 10 × 2.5 = Rs. 25.00

Programme variable	*Cost per unit in Rs.*	*Cost requirement* $\xrightarrow{C_j}$	3 *x*	2.5 *y*	0 *p*	0 *q*	*M* A_1	*M* A_2	*Replacement ratio*
y	2.5	10	0.5	1	– 0.25	0	0.25	0	10/0.5 = 20
A_2	*M*	30	**2**	0	0.5	– 1	–0.5	1	30/2 = 15

Z_j			1.25 + 2 M	2.5	0.5 M – 0.625	– M	0.625 – 0.5 M	0	
$C_j – Z_j$			1.75 – 2 M	0	0.625 – .5 M	M	1.5 M – 0.625	0	

Changing the key row: 30/2 = 15, 2/2 = 1, 0/2 = 0, 0.5/2 = 0.25, –1/2 = –0.5, –0.5/2 = –0.25, 1/2 = 0.5.

Changing the non key row:

10 – 30 × 0.5/2 = 2.5

0.5 – 2 × 0.25 = 0

1 – 0 × 0.25 = 1

–0.25 – 0.5 × 0.25 = –0.375

0 – (–1) × 0.25 = 0.25

0.25 – (–0.5) × 0.25 = 0.375

0 – 1 × 0.25 = – 0.25

Entering the above in the simplex table 3.

Table: 3.

x = 15, y = 2.5, p = 0, q = 0 , A_1 = 0, A_2 = 0 and Z = Rs. 15 × 3 + Rs. 2.5 × 2.5 = 45 + 6.25 =
Rs. 51.25

Programme Variable	*Cost per unit in Rs.*	*Cost* → C_j *Requirement*	*3* x	*2.5* y	*0* p	*0* q	M A_1	M A_2	*Replacement ratio*
y	*2.5*	2.5	0	1	– 0.375	0.25	0.375	– 0.25	—
A_x	3	15	1	0	0.25	–0.5	–0.25	0.5	—
Z_j			3	2.5	– 0.188	0.875	0.188	0.875	
$C_j – Z_j$			0	0	0.188	0.875	M – 0.188	M– 0.875	

Optimal Cost = Z^* = 3 × 15 + 2.5 × 2.5 = 45 + 6.25 = Rs. 51.25

Imputed value = 0.1875 × 40 + 0.875 × 50 = 7.5 + 43.75 = Rs. 51.25.

As all the elements of net evaluation row are either zeros or positive elements the solution is optimal.

The patient has to purchase 15 units of X and 2.5 units of Y to meet the requirement and the cost is Rs. 51.25/– While solving maximisation problem, we have seen that the elements in net evaluation row, *i.e.*, ($C_j – Z_j$) row associated with slack variables represent

***Point to Note*:**

1. **In the mechanics of simplex method of minimization problem, once an artificial surplus variable leaves the basis, its exit is final because of its high cost coefficient (M), which will never permit the variable to reenter the basis. In order to save time or to reduce calculations, we can cross out the column containing the artificial surplus variable, which reduces the number of columns.**
2. **A better and easier method is to allocate a value for M in big M method; this value must be higher than the cost coefficients of the decision variables. Say for example the cost coefficients of the decision variable in the above problem are for X it is Rs.3/– and for Y it is Rs. 2.5. We can allocate a cost coefficient to M as Rs.10, which is greater than Rs.3/– and Rs. 2.5. Depending the value of decision variables, this value may be fixed at a higher level (Preferably the value must be multiples of 10 so that the calculation part will be easier.**

the marginal worth or shadow price of the resources. In minimisation problem, the elements associated with surplus variables in the optimal table, represent the marginal worth or imputed value of one unit of the required item. **In minimisation problem, the imputed value of surplus variables in optimal solution must be equal to the optimal cost.**

WORKED OUT PROBLEMS

Example . A company manufactures two products *X* and *Y* whose profit contributions are Rs. 10 and Rs. 20 respectively. Product *X* requires 5 hours on machine I, 3 hours on machine II and 2 hours on machine III. The requirement of product *Y* is 3 hours on machine I, 6 hours on machine II and 5 hours on machine III. The available capacities for the planning period for machine I, II and III are 30, 36 and 20 hours respectively. Find the optimal product mix.

Solution: The given data:

Machine	*Products (Time required in hours)*		*Availability in hours*
	X	*Y*	
I	5	3	30
II	3	6	36
III	2	5	20
Profit per unit in Rs.	10	20	

Inequalities:

Maximize $Z = 10x + 20y$ s.t.

s.t.

$5x + 3y \leq 30$

$3x + 6y \leq 36$

$2x + 5y \leq 20$ and

Both x and y are ≥ 0.

Simplex format:

Maximize $Z = 10x + 20y + 0S_1 + 0S_2 + 0S_3$

$5x + 3y + 1S_1 + 0S_2 + 0S_3 = 30$

$3x + 6y + 0S_1 + 1S_2 + 0S_3 = 36$

$2x + 5y + 0S_1 + 0S_2 + 1S_3 = 20$ and

x, y, S_1, S_2 and S_3 all ≥ 0.

Table: I. $x = 0$, $y = 0$ $S_1 = 30$, $S_2 = 36$ and $S_3 = 20$, $Z =$ Rs. 0

Problem variable	*Profit in Rs.*	*Capacity*	$C_j = 10$ x	20 y	0 S_1	0 S_2	0 S_3	*Replacement ratio*
S_1	0	30	5	3	1	0	0	30/3 = 10
S_2	0	36	3	6	0	1	0	36/6 = 6
S_3	0	20	2	**5**	0	0	1	20/5 = 4
Opportunity cost.			10	20	0	0	0	

Step 1: For the first table the net evaluation row elements are same as profit row element. For successive rows, the key column element is given by Profit – (Sum of key column element x respective object row element). For maximization problem, select the highest element among the key column element and mark an arrow as shown to indicate incoming variable. For minimization problems select the lowest element or highest element with negative sign and write an arrow to indicate the incoming variable. Enclose key column elements in a rectangle. Here they are shown in red colour. It is known as key row because it gives the clue about incoming variable.

Step 2: Divide the capacity column numbers with respective key column number to get the replacement ratio. Select the lowest ratio as the indicator of out going variable. The lowest ratio is also known as limiting ratio. In the above table the limiting ratio elements are 10, 6, 4. We select 4 as the indicator of outgoing variable. It is because, if we select any other number the third machine cannot compete more than 4 units of product. Though the machine has got capacity to manufacture 10 units and second machine has got capacity of 6 units, only 4 units can be manufactured completely. Rest of the capacity of machine 1 and 2 will become idle resource. Enclose the elements of key row in a rectangle. It is known as key row because it gives the clue about out going variable. Mark this row with a tick mark. (Here the elements are marked in thick.

Step 3: The element at the intersection of key row and key column is known as Key number, in the table it is marked in bold thick number. It is known as the key number, because, the next table is written with the help of this key element.

Table: II. $x = 0$, $y = 4$, $S_1 = 18$, $S_2 = 12$, $S_3 = 0$. Z = Rs. 4 × 20 = Rs. 80.

Problem variable	*Profit in Rs.*	*Capacity*	*C_j = 10 x*	*20 y*	*0 S_1*	*0 S_2*	*0 S_3*	*Replacement ratio*
S_1	0	18	**3.8**	0	1	0	– 0.6	18/3.8 = 4.8
S_2	0	12	0.6	0	0	1	– 1.2	12/0.6 = 2.0
y	20	4	0.4	1	0	0	0.2	4/0.4 = 10
Opportunity cost.			2	0	0	0	– 4	

Step 4: To improve the programme or to get the new table the following procedure is adopted:

i) Transfer of key row of old tableau: Divide all the elements of key row of old tableau by key number, which gives the key row elements of the new tableau.

ii) To transfer non key rows: New row number = old row number – (corresponding key row number × fixed ratio.)

Here, fixed ratio = (key column number of the row/key number).

While transferring remembers you should not alter the positions of the rows. Only incoming variable replaces the outgoing slack variable or any other outgoing basis variable as the case may be.

The net evaluation row element of the variable entered into the programme will be zero. When all the variables are transferred, the identity matrix will come in the position of main matrix.

To check whether, the problem is done in a correct manner, check that whether profit earned at present stage is equal to shadow price at that stage. Multiplying the net evaluation row element under non-basis can get shadow price variable (identity matrix) by original capacities of resources given in the problem.

Above explained procedure of transferring key row and non-key rows is worked out below: Transfer of Key row: 20 / 5 = 4, 2 / 5 = 0.4, 5 / 5 = 1, 0 /5 = 0, 0 / 5 = 0, and 1 / 5 = 0.2.

Transfer of non-key rows:

30 – 20 × 3/5 = 18	36 – 20 × 6 / 5 = 12
5 – 2 × 3/5 = 3.8	3 – 2 × 1.2 = 0.6
3 – 5 = 3/5 = 0	6 – 5 × 1,2 = 0
1 – 0 × 3/5 = 1	0 – 0 × 1.2 = 0
0 – 0 × 3/5 = 0	1 – 0 × 1.2 = 1

$0 - 1 \times 3/5 = -0.6$ $\qquad$ $0 - 1 \times 1.2 = -1.2$

Step 5: Once the elements of net evaluation row are either negatives or zeros (for maximization problem), the solution is said to be optimal. For minimization problem, the net evaluation row elements are either positive elements or zeros.

As all the elements of net evaluation row are either zeros or negative elements, the solution is optimal. The company will produce 4.8 units of *X* and 3.6 units of *Y* and earn a profit of Rs. 120/–.

Shadow price is $2.6 \times 30 + 2 \times 20 =$ Rs. 128/–. The difference of Rs. 8/– is due to decimal values, which are rounded off to nearest whole number.

Table: III. $x = 4.8$, $y = 3.6$, $S_2 = 9$, $S_1 = 0$, $S_3 = 0$ and $Z = 4.8 \times 10 + 3.6 \times 20 =$ Rs. 120/–

Problem variable	*Profit in Rs.*	*Capacity*	C_j = 10 *x*	20 *y*	0 S_1	0 S_2	0 S_3	*Replacement ratio*
x	10	4.8	1	0	0.26	0	– 0.16	
S_2	0	9	0	0	– 0.16	1	– 1.1	
y	20	3.6	0	1	0	0	0.18	
Opportunity cost.			0	0	– 2.6	0	–2.0	

↑

Problem. 4: A company manufactures three products namely *X*, *Y* and *Z*. Each of the product require processing on three machines, Turning, Milling and Grinding. Product *X* requires 10 hours of turning, 5 hours of milling and 1 hour of grinding. Product *Y* requires 5 hours of turning, 10 hours of milling and 1 hour of grinding, and Product Z requires 2 hours of turning, 4 hours of milling and 2 hours of grinding. In the coming planning period, 2700 hours of turning, 2200 hours of milling and 500 hours of grinding are available. The profit contribution of *X*, *Y* and Z are Rs. 10, Rs.15 and Rs. 20 per unit respectively. Find the optimal product mix to maximize the profit.

Solution: The given data can be written in a table.

Machine	*Product* *Time required in hours per unit*			*Available hours*
	X	*Y*	*X*	
Turning.	10	5	2	2,700
Milling	5	10	4	2,200
Grinding.	1	1	2	500

Profit contribution in Rs. per unit.	10	15	20	

Let the company manufacture x units of X, y units of Y and z units of Z

Inequalities:

Maximise $Z = 10x + 15y + 20z$ S.T.

S.T $10x + 5y + 2z \leq 2{,}700$

$5x + 10y + 4z \leq 2{,}200$

$1x + 1y + 2z \leq 500$ and

All x, y and z are ≥ 0

Equations:

Maximise $Z = 10x + 15y + 20x$

$10x + 5y + 2z + 1S_1 = 2700$

$5x + 10y + 4z + 1S_2 = 2200$

$1x + 1y + 2z + 1S_3 = 500$ and

x, y and z all ≥ 0

Simplex format:

Maximise $Z = 10x + 15y + 20z + 0S_1 + 0S_2 + 0S_3$ S.t. $10x + 5y + 2z + 1S_1 + 0S_2 + 0S_3 = 2700$

$5x + 10y + 4z + 0S_1 + 1S_2 + 0S_3 = 2200$

$1x + 1y + 2z + 0S_1 + 0S_2 + 1S_3 = 500$

And all x, y, z, S_1, S_2, S_3 are ≥ 0

Table I. $x = 0$, $y = 0$, $z = 0$, $S_1 = 2700$, $S_2 = 2200$, $S_3 = 500$. Profit Z = Rs. 0

Programme	*Profit*	*Capacity*	$C_j=10$ x	15 y	20 z	0 S_1	0 S_2	0 S_3	*Replacement ratio*	*Check column.*
S_1	0	2700	10	5	2	1	0	0	2700/2 = 13500	2718
S_2	0	2200	5	10	4	0	1	0	2200/4 = 550	2220
S_3	0	500	1	1	**2**	0	0	1	500/2 = 250	505
Net evaluation			10	15	20	0	0	0		

{Note: The check column is used to check the correctness of arithmetic calculations. The check column elements are obtained by adding the elements of the corresponding row starting from capacity column to the last column (avoid the elements of replacement ration column). As far as treatment of check column is concerned it is treated on par with elements in other columns. In the first table add the elements of the row as said above and write the elements of check column. In the second table onwards, the elements are got by usual calculations. Once you get elements, add the elements of respective row starting from capacity column to the last column of identity, then that sum must be equal to the check column element.}

Table: II. $x = 0$, $y = 0$, $z = 250$ units, $S_1 = 2200$, $S_2 = 1200$, $S_3 = 0$ and Z = Rs. 20×250 = Rs. 5,000.

Programme	*Profit*	*Capacity*	*C_j=10 x*	*15 y*	*20 z*	*0 S_1*	*0 S_2*	*0 S_3*	*Check column.*	*Replacement ratio*
S_1	0	2210	9	4	0	1	0	– 1	2213	552.5
S_2	0	1200	3	**8**	0	0	1	– 2	1210	150
Z	20	250	0.5	0.5	1	0	0	0.5	500	500
Net Evaluation.			0	5	0	0	0	– 10		

Profit at this stage = Rs. 20 × 250 = Rs. 5,000 and Shadow price = 10 × 500 = Rs. 5000.

Table: III. $x = 0$, $y = 150$, $z = 174.4$, $S_1 = 1600$, $S_2 = 0$, $S_3 = 0$ and Z = Rs. 5738/–

Programme	*Profit*	*Capacity*	*C_j=10 x*	*15 y*	*20 z*	*0 S_1*	*0 S_2*	*0 S_3*	*Check column.*	*Replacement Ratio*
S_1	0	1600	7.5	0	0	1	–0.5	0	1608	
Y	15	150	0.375	1	0	0	0.125	–0.25	151.25	
Z	20	174.4	0.311	0	1	0	– 0.063	0.626	423.7	
Net Evn.			–1.85	0	0	0	– 0.615	– 8.77		

As all the elements of Net evaluation row are either zeros or negative elements, the solution is optimal. The firm has to produce 150 units of Y and 174.4 units of *Z*. The optimal profit = 15 × 150 + 20 × 174.4 = Rs.5738 /–

To check the shadow price = 0.615 × 2200 + – 8.77 × 500 = 1353 + 4385 = Rs. 5738 /–.

MINIMISATION PROBLEMS

Problem 5: A small city of 15,000 people requires an average of 3 lakhs of gallons of water daily. The city is supplied with water purified at a central water works, where water is purified by filtration, chlorination and addition of two chemicals softening chemical *X* and health chemical *Y*. Water works plans to purchase two popular brands of products, product *A* and product *B*, which contain these two elements. One unit of product A gives 8 Kg of *X* and 3 Kg of *Y*. One unit of product B gives 4 Kg of *X* and 9 Kg of *Y*. To maintain the water at a minimum level of softness and meet a minimum in health protection, it is decided that 150 Kg and 100 Kg of two chemicals that make up each product must be added daily. At a cost of Rs. 8/– and Rs. 10/– per unit respectively for A and B, what is the optimum quantity of each product that should be used to meet consumer standard?

Before discussing solution, let us have an idea of what is known as **Big M–n Method**, which is generally used to solve minimization problems.

While solving the linear programming problems by graphical method, we have seen an isoprofit line is drawn and at the origin and then it is moved away from the origin to find the optima point. Similarly an isocost line is drawn away from the origin in minimization problem and moved towards the origin to find the optimal point.

But in simplex method of solving the minimization problem, a highest cost is allocated to artificial surplus variable to remove it form the matrix. This high cost is Big – *M*. *M* stands for millions of rupees. If we use big *M* some times we feel it difficult while solving the problem. Hence, we can substitute a big numerical number to *M*, which is bigger than all the cost coefficients given in the problem. This may help us in numerical calculations.

Solution: Let the water works purchase x units of X and y units of Y, then:

Inequalities:

Minimise $Z = 8x + 10y$ s.t.

$3x + 9y \geq 100$

$8x + 4y \geq 150$ and

Both x and $y \geq 0$

Simplex Format:

Minimise $Z = 8x + 10y + 0p + 0q + M A_1 + MA_2$ s.t.

$3x + 9y - 1p + 0q + 1A_1 + 0A_2 = 100$

$8x + 4y + 0p - 1q + 0A_1 + 1A_2 = 150$ and

x, y, p, q, A_1, A_2 all ≥ 0

Table: I. $x = 0$, $y = 0$, $p = 0$, $a = 0$, $A_1 = 100$, $A_2 = 150$ and $Z = 100M + 150M =$ Rs. 250 M.

Programe	*Cost in Rs.*	*C_j = requirement*	*8* *x*	*10* *y*	*0* *p*	*0* *q*	*M* *A_1*	*M* *A_2*	*Replacement ratio*
A_1	M	100	3	**9**	–1	0	1	0	100/9 = 11.11
A_2	M	150	8	4	0	–1	0	1	150 / 4 =

									37.5
Net	Evaluation		8 – 11M	10 – 13M	M	M	0	0	—

Table II: $x = 0$, $y = 11.11$, $p = 0$, $q = 0$, $A_1 = 1.32$, $A_2 = 0$, $Z =$ Rs. 11.11 x 10 + 1.32M =111.1 +1.32 M

Program	*Cost inRs.*	*C_j = requirement*	*8 x*	*10 y*	*0 p*	*0 q*	*M A_1*	*M A_2*	*Replacement ratio*
y	10	11.1	0.33	1	–0.11	0	0.11	0	33.6
A_2	M	106	**6.88**	0	0.44	–1	–0.44	1	15.4
Net	Evaluation		4.3–6.88M	0	–1.1+0.44M	M	–1.1+5.4 M	0	

Table III. $x = 0.5$, $y = 15.4$, $p = 0$, $q = 0$, $A_1 = 0$, $A_2 = 0$, $Z =$ Rs. 10 x 0.50 + 8 x 15.4 = Rs. 128.20

Program	*Cost inRs.*	*C_j = requirement*	*8 x*	*10 y*	*0 p*	*0 q*	*M A_1*	*M A_2*	*Replacement ratio*
y	10	0.5	0	1	–0.154	0.1	0.154	–0.1	—
x	8	15.4	1	0	0.06	–0.14	–0.06	0.14	—
Net	Evaluation	—	0	0	1.062	0.12	M–1.06	M–0.12	

Water works purchases 0.5 Kg of *Y* and 15.4 Kg of *X* at a cost of Rs. 128.20. The shadow price will be Rs. 107/–. The difference is due to decimal numbers. (Note: We can avoid the artificial variables as and when they go out to reduce the calculations. We can use a numerical value for *M*, which is higher than the cost of variables given in the problem so that we can save time.).

Problem 6: 10 grams of Alloy A contains 2 grams of copper, 1 gram of zinc and 1 gram of lead. 10 grams of Alloy B contains 1 gram of copper, 1 gram of zinc and 1 gram of lead. It is required to produce a mixture of these alloys, which contains at least 10 grams of copper, 8 grams of zinc, and 12 grams of lead. Alloy B costs 1.5 times as much per Kg as alloy A. Find the amounts of alloys *A* and *B*, which must be mixed in order to satisfy these conditions in the cheapest way.

Solution: The given data is: (Assume the cost of Alloy *A* as Re.1/– then the cost of Alloy *B* willbe Rs. 1.50 per Kg.

Metals	*Alloys (In grams per 10 grams)*		*Requirement in Grams*
	A	*B*	
Copper	2	1	10
Zinc	1	1	8
Lead	1	1	12
Cost in Rs. per Kg.	1	1.5	

Let the company purchase *x* units of Alloy *A* and *y* units of Alloy *B*. (Assume a value of 10 for *M*)

Inequalities:	Simplex Format:
Minimise Z = $1x + 1.5y$ s.t.	Minimise $Z = 1x + 1.5y + 0p + 0q + 0r + 10A_1$
$+ 10A_2 + 10A_3$ s.t. $2x + 1y \geq 10$	$2x + 1y - 1p + 0q + 0r + 1A_1 + 0A_2 + 0A_3 = 10$
$1x + 1y \geq 8$	$1x + 1y + 0p - 1q + 0r + 0A_1 + 1A_2 + 0A_3 = 8$
$1x + 1y \geq 12$ and and	$1x + 1y + 0p + 0q - 1r + 0A_1 + 0A_2 + 1A_3 = 12$
x, y both ≥ 0	$x, y, p, q, r, A_1, A_2, A_3$ all ≥ 0

Table: I. $x = 0$, $y = 0$, $p = 0$, $q = 0$, $r = 0$, $A_1 = 10$, A_2 8 and $A_3 = 12$ and Profit Z = Rs. $10 \times 10 + 10 \times 8 + 10 \times 12$ = Rs. 300

Program	*Cost in Rs.*	*C_j = Requir e–ment*	*1 x*	*1.5 y*	*0 p*	*0 q*	*0 r*	*10 A_1*	*10 A_2*	*10 A_3*	*Replace – ment ratio*
A_1	10	10	**2**	1	–1	0	0	1	0	0	5
A_2	10	8	1	1	0	–1	0	0	1	0	8
A_3	10	12	1	1	0	0	–1	0	0	1	12
Net	Evaluation		–39	–28.5	10	10	10	0	0	0	

Table: II. $x = 5$, $y = 0$, $p = 0$, $q = 0$, $r = 0$, $A_1 = 0$, $A_2 = 3$, $A_3 = 7$ and Z = Rs. 1 × 5 + 10 x 3 + 7 ×10 = Rs. 105/–

Program	*Cost in Rs.*	*C_j = Requir e–ment*	*1 x*	*1.5 y*	*0 p*	*0 q*	*0 r*	*10 A_1*	*10 A_2*	*10 A_3*	*Replace – ment ratio*
x	1	5	1	0.5	– 0.5	0	0	0.5	0	0	– 10 (neglect)
A_2	10	3	0	0.5	**0.5**	– 1	0	– 0.5	1	0	6
A_3	10	7	0	0.5	0.5	0	– 1	– 0.5	0	1	14
Net	Evaluation		0	– 9	–9.5	10	10	19.5	0	0	

Table: III. $x = 8$, $y = 0$, $p = 6$, $q = 0$, $r = 0$, $A_1 = 0$, $A_2 = 0$, $A_3 = 4$, Z = Rs. 8 + 40 = Rs. 48 /–

Program	*Cost in Rs.*	*C_j = Requir e–ment*	*1 x*	*1.5 y*	*0 p*	*0 q*	*0 r*	*10 A_1*	*10 A_2*	*10 A_3*	*Replace – ment ratio*
x	1	8	1	1	0	– 1	0	0	1	0	– 8 (neglect)
p	0	6	0	1	1	– 2	0	– 1	2	0	– 3 (neglect)
A_3	10	4	0	0	0	**1**	– 1	0	–1	1	4
Net	Evaluation		0	0.5	0	– 9	10	10	19	0	

Table: IV. $x = 12$, $y = 0$, $p = 14$, $q = 4$, $r = 0$, Z = Rs. 1 × 12 = Rs. 12/ –

Program	*Cost in Rs.*	*C_j = Requir e–ment*	*1 x*	*1.5 y*	*0 p*	*0 q*	*0 r*	*10 A_1*	*10 A_2*	*10 A_3*	*Replace – ment ratio*
x	1	12	1	1	0	0	–1	0	0	1	

p	0	14	0	1	1	0	–2	–1	1	2	
q	0	4	0	0	0	1	–1	0	–1	1	
Net	Evaluation		0	0.5	0	0	1	0	0	9	

As all the net evaluation elements are either zeros or positive element, the solution is optimal. The company can purchase 12 units of X at a cost of Rs. 12/–

Problem 3.12: Minimise $Z = 4a + 2b$ s.t. $3a + 1b$ □ 27

$–1a – 1b$ □ –21

$1a + 2b$ □ 30 and both a and b are □ 0.

Inequalities: Equations:

Minimise $Z = 4a + 2b \geq 27$ Minimise $Z = 4a + 2b + 0p + 0q + 0r + MA_1 + MA_2 + MA_3$ s.t. $3a + 1b$ □ 27 $3a + 1b – 1p + 0q + 0r + 1A_1 + 0A_2 + 0A_3 = 27$

$1a + 1b$ □ 21 $1a + 1b + 0p – 1q + 0r + 0A_1 + 1A_2 + 0A_3 = 21$

$1a + 2b$ □ 30 $1a + 2b + 0p$ +– $+ 0q – 1r + 0A_1 + 0A_2 + 1A_3 = 30$

(**Note: converting the objective function conveniently we can solve the minimization or maximization problems.** For example, if the objective function given is minimization type, we can convert it into maximization type by multiplying the objective function by –1. For example, in the problem 3.12, the objective function may be written as Maximise $Z = –4a – 2b$ s.t. But the inequalities are in the form of ≥ type. In such cases when artificial surplus variable (A_I) is introduced then the cost co–efficient of the artificial surplus variable will be $–M$ instead of $+M$. Rest of the procedure of solving the problem is same. Similarly, any maximization problem can be converted into minimization problem by multiplying the objective function by –1. If the inequalities are in ≥ form, subtracting the surplus variable and adding the artificial surplus variable is done to inequalities to convert them into equations. In case the inequalities are of ≤ type, slack variable is added to convert them into equations. Let us see this in next example).

And a, b both □ 0 a, b, p, q, r, A_1, A_2 and A_3 all □ 0

Solution: Let M be represented by a numerical value Rs.10/– that is higher than the cost coefficients given in the problem. (*i.e*. 4 and 2).

Table: I. $a = 0$, $b = 0$, $p = 0$, $q = 0$, $r = 0$, $A_1 = 27$, $A_2 = 21$ and $A_3 = 30$ and the cost Z = Rs. 780/–

Program	Cost in Rs.	*Cj* Requirement	*4* *a*	*2* *b*	*0* *p*	*0* *q*	*0* *r*	*10* A_1	*10* A_2	*10* A_3	Replacement ratio
A_1	10	27	**3**	1	– 1	0	0	1	0	0	9
A_2	10	21	1	1	0	– 1	0	0	1	0	21
A_3	10	30	1	2	0	0	– 1	0	0	1	30
	Net	Evaluation	– 46	– 38	10	10	10	0	0	0	—

Table: II. $a = 9$, $b = 0$, $p = 0$, $q = 0$, $r = 0$, $A_1 = 0$, $A_2 = 12$, $A_3 = 21$ and Z = Rs. 366/–

Program	Cost in Rs.	*Cj* Requirement	*4* *a*	*2* *b*	*0* *p*	*0* *q*	*0* *r*	*10* A_1	*10* A_2	*10* A_3	Replacement ratio
a	4	9	1	0.33	– 0.33	0	0	0.33	0	0	27.27
A_2	10	12	0	0.67	0.33	– 1	0	– 0.33	1	0	17.91
A_3	10	21	0	**1.67**	0	0	– 1	0	0	1	12.51
	Net	Evaluation	0	– 22.72	– 1.98	10	10	8.22	0	0	

Table: III. $a = 4.84$, $b = 12.51$, $p = 0$, $q = 0$, $r = 0$, $A_1 = 0$, $A_2 = 3.6$, $A_3 = 0$, Z = Rs. 80.50.

Program	Cost in Rs.	*Cj* Requirement	*4* *a*	*2* *b*	*0* *p*	*0* *q*	*0* *r*	*10* A_1	*10* A_2	*10* A_3	Replacement ratio
a	4	4.84	1	0	– 0.33	0	– 0.198	0.33	0	0.198	Neglect
A_2	10	3.6	0	0	0.33	– 1	– 0.4	– 0.33	**1**	0.4	10.9
b	2	12.57	0	1	0	0	– 0.6	0	0	0.6	Infinity
	Net	Evaluation	0	0	– 1.98	10	3.592	8.02	0	4.008	—

Table: IV. $a = 3$, $b = 18.05$, $p = 9$, $q = 0$, $r = 0$, $A_1 = 0$, $A_2 = 0$, $A_3 = 0$, Cost Z = Rs. 48.10

Program	*Cost in Rs.*	*Cj requirement*	*4 x*	*2 y*	*0 p*	*0 q*	*0 r*	*10 A_1*	*10 A_2*	*10 A_3*	*Replacement ratio*
a	4	3	1	0	–0.485	0.5	0	0.485	–0.5	0	
p	0	9	0	0	0.425	–2.5	1	–0.425	0.25	–1	
b	2	18.05	0	1	0.445	–1.5	0	–0.445	1.5	0	
	Net	Evaluation	0	0	1.05	1	0	8.95	9	0	

As the elements of net evaluation row are either zeros or positive elements, the solution is optimal.

$A = 3$ and $B = 12.57$ and the optima cost Z = Rs. 48.10. The shadow price = $1.05 \times 27 + 1 \times 21$ = Rs. 49.35.

The difference is due to decimal calculations.

Problem 7: Solve the Minimization L.P.P. given below:

Min. $Z = 1x - 3y + 2z$

S.t. $3x - 1y - + 3z$ □ 7

$-2x + 4y + 0z$ □ 12

$-4x + 3y + 8z$ □ 10 and x, y, and z all □ 0.

Solution: As the objective function is of minimization type and the constraints are of □ type, wecan rewrite the problem in simplex format as:

Maximize $Z = -1x + 3y - 2z + 0S_1 + 0S_2 + 0S_3$ S.t. $3x - 1y + 3z + 1S_1 + 0S_2 + 0S_3 = 7$

$-2x + 4y + 0z + 0S_1 + 1S_2 + 0S_3 = 12$

$-4x + 3y + 8z + 0S_1 + 0S_2 + 1S_3 = 10$ and x, y, z, S_1, S_2 and S_3 all ≥ 0.

Table: I. $x = 0$, $y = 0$, $z = 0$, $S_1 = 7$, $S_2 = 0$, $S_3 = 0$ and Z = Rs.0./–

Problem variable	*Profit Rs.*	*Capacity, = Z units*	*– 1 x*	*3 y*	*– 2 z*	*0 S_1*	*0 S_2*	*0 S_3*	*Replacement ratio*
S_1	0	7	3	– 1	3	1	0	0	—
S_2	0	12	– 2	**4**	0	0	1	0	12/4 = 3
S_3	0	10	– 4	3	8	0	0	1	10/3 = 3.3.
	Net	Evaluation.	– 1	3	– 2	0	0	0	

Table: II. $x = 0$, $y = 3$, $z = 0$, $S_1 = 10$, $S_2 = 0$, $S_3 = 1$ and $Z = 3 \times 3 =$ Rs.9.00.

Problem variable	*Profit Rs.*	*Capacity, Z = units*	*– 1 x*	*3 y*	*– 2 z*	*0 S_1*	*0 S_2*	*0 S_3*	*Replacement ratio*
S_1	0	10	**3.5**	0	3	1	0.25	0	2.86
y	3	3	– 0.5	1	0	0	0.25	0	—
S_3	0	1	– 2.5	0	8	0	– 0.75	1	—
	Net	Evaluation.	0.5	0	– 2	0	– 0.75	0	

Table III. $x = 2.86$, $y = 4.43$, $z = 0$, $S_1 = 0$, $S_2 = 0$ and $S_3 = 8.14$, $Z =$ Rs. 10.43.

Problem variable	*Profit Rs.*	*Capacity, Z = units*	*– 1 x*	*3 y*	*– 2 z*	*0 S_1*	*0 S_2*	*0 S_3*	*Replacement ratio*
x	– 1	2.86	1	0	0.86	0.29	0.07	0	—
y	3	4.43	0	1	0.43	0.143	0.285	0	—
S_3	0	8.14	0	0	10.14	0.174	– 0.57	1	—
	Net	Evaluation.	0	0	– 2.43	– 0.14	– 0.68	0	

Answer: $X = 2.86$, $Y = 4.43$, $Z = 0$ and Profit $Z =$ Rs. 10.43.

UNIT – II TRANSPORTATION

INTRODUCTION

In operations Research Linear programming is one of the model in mathematical programming, which is very broad and vast. Mathematical programming includes many more optimization models known as Non - linear Programming, Stochastic programming, Integer Programming and Dynamic Programming
- each one of them is an efficient optimization technique to solve the problem with a specific structure, which depends on the assumptions made in formulating the model. We can remember that the general linear programming model is based on the assumptions:

(a) Certainty: The resources available and the requirement of resources by competing candidates, the profit coefficients of each variable are assumed to remain unchanged and they are certain in nature.

(b) Linearity: The objective function and structural constraints are assumed to be linear.

(c) Divisibility: All variables are assumed to be continuous; hence they can assume integer or fractional values.

(d) Single stage: The model is static and constrained to one decision only. And planning period is assumed to be
fixed.

(e) Non-negativity: A non-negativity constraint exists in the problem, so that the values of all variables are to be $\geq$ 0,
i.e. the lower limit is zero and the upper limit may be any positive number.

(f) Fixed technology: Production requirements are assumed to be fixed during the planning period.

(g) Constant profit or cost per unit: Regardless of the production schedules profit or cost remain constant.

Now let us examine the applicability of linear programming model for transportation and assignment models.

TRANSPORTATION MODEL

The transportation model deals with a special class of linear programming problem in which the objective is to **transport a homogeneous commodity from various origins or factories to different destinations or markets at a total minimum cost.**

To understand the problem more clearly, let us take an example and discuss the rationale of transportation problem. Three factories *A*, *B* and *C* manufactures sugar and are located in different regions. Factory *A* manufactures, b_1 tons of sugar per year and *B* manufactures b_2 tons of sugar per year and *C* manufactures b_3 tons of sugar. The sugar is required by four markets *W*, *X*, *Y* and *Z*. The requirement of the four markets is as follows: Demand for sugar in Markets *W*, *X*, Yand *Z* is d_1, d_2, d_3 and d_4 tons respectively. The transportation cost of one ton of sugar from each factory to market is given in the matrix below. The objective is to transport sugar from factories to the markets at a minimum total transportation cost.

	Markets	Transportation cost per ton in Rs.				Availability in tons
		W	*X*	*Y*	*Z*	
Factories	*A*	c_{11}	c_{12}	c_{13}	c_{14}	b_1
	B	c_{21}	c_{22}	c_{23}	c_{24}	b_2
Demand in Tons.	*C*	c_{31}	c_{32}	c_{33}	c_{34}	b_3
		d_1	d_2	d_3	d_4	$\square b_j / \square d_j$

For the data given above, the mathematical model will be:

Minimize $Z = c_{11}\,x_{11} + c_{12}\,x_{12} + c_{13}\,x_{13} + c_{14}\,x_{14} + c_{21}\,x_{21} + c_{22}\,x_{22} + c_{23}\,x_{23} + c_{24}\,x_{24} +$

$c_{31}\,x_{31} + c_{32}\,x_{32} + c_{33}\,x_{33} + c_{34}\,x_{34}$ subject to a condition: $\longrightarrow$ **OBJECTIVE FUNCTION.**

$a_{11}\,x_{11} + a_{12}\,x_{12} + a_{13}\,x_{13} + a_{14}\,x_{14} \leq b_1$ (because the sum must be less than or equal to the available capacity)

$a_{21}\,x_{21} + a_{22}\,x_{22} + a_{23}\,x_{23} + a_{24}\,x_{24} \leq b_2$ $a_{31}\,x_{31} + a_{32}\,x_{32} + a_{33}\,x_{33} + a_{34}\,x_{34} \leq b_3$ $\longrightarrow$ **MIXED STRUCTURAL CONSTRAINTS.**

$a_{11}\,x_{11} + a_{21}\,x_{21} + a_{31}\,x_{31} \geq d_1$

$a_{12}\,x_{12} + a_{22}\,x_{22} + a_{32}\,x_{32} \geq d_2$ $a_{13}\,x_{13} + a_{23}\,x_{23} + a_{33}\,x_{33} \geq d_3$ (because the sum must be greater than or equal to the demand of the market. We cannot send less than what is required)

$a_{14}\, x_{14} + a_{24}\, x_{24} + a_{34}\, x_{34} \geq d_4$ and

All x_{ij} and x_{ji} are ≥ 0 where $i = 1,2,3$ and $j = 1,2,3,4$. (This is because we cannot supply negative elements).

NON-NEGATIVITY CONSTRAINT.

The above problem has got the following properties:

1. It has an objective function.
2. It has structural constraints.
3. It has a non-negativity constraint.
4. The relationship between the variables and the constraints are linear.

We know very well that these are the properties of **a linear programming problem**. Hence the transportation model is also a **linear programming problem**. But a **special type of linear programming problem.**

Once we say that the problem has got the characteristics of linear programming model, and then we can solve it by simplex method. Hence we can solve the transportation problem by using the simplex method. As we see in the above given transportation model, the structural constraints are of mixed type. That is some of them are of $\leq$ type and some of them are of $\geq$ type. When we start solving the transportation problem by simplex method, it takes more time and laborious. Hence we use **transportation algorithm or transportation method** to solve the problem. Before we discuss the transportation algorithm, let us see how a general model for transportation problem appears. The general problem will have '*m*' rows and '*n*' columns *i.e.*, $m \times n$ matrix.

Minimize $Z = \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij}\, x_j$ s.t. where $i = 1$ to m and $j = 1$ to n.

$\sum_{i=1}^{m} a_{ij}\, x_{ij} \;\square\; b_i$ where $i = 1$ to m and $j = 1$ to n

$\sum_{j=1}^{n} a_{ij}\, x_{ji} \;\square\; d_j$ where $i = 1$ to m and $j = 1$ to n

Comparison Between Transportation Model and General LinearProgramming Model

Differences

1. Transportation model is basically a minimization model; where as general linear programmingmodel may be of maximization type or minimization type.
2. The resources, for which, the structural constraints are built up is homogeneous in transportation model; where as in general linear programming model they are different. That is one of the constraint may relate to machine hours and next one may relate to man-hours etc. In transportation problem, all the constraints are related to one particular resource or commodity, which is manufactured by the factories and demanded by the market points.
3. The transportation problem is solved by transportation algorithm; where as the general linear programming problem is solved by simplex method.
4. The values of structural coefficients (*i.e.* x_{ij}) are not restricted to any value in general linear programming model, where as it is restricted to values either 0 or 1 in transportation problem.Say for example:

   Let one of the constraints in general linear programming model is: $2x - 3y + 10z \leq 20$. Here the coefficients of structural variables x, y and z may negative numbers or positive numbers of zeros. Where as in transportation model, say for example $x_{11} + x_{12} + x_{13} + x_{14} = b_i = 20$. Suppose the value of variables x_{11}, and x_{14} are 10 each, then $10 + 0.\ x_{12} + 0.\ x_{13} + 10 = 20$. Hence the coefficients of x_{11} and x_{14} are 1 and that of x_{12} and x_{13} are zero.

Similarities

1. Both have objective function.
2. Both have linear objective function.
3. Both have non - negativity constraints.
4. Both can be solved by simplex method. In transportation model it is laborious.
5. A general linear programming problem can be reduced to a transportation problem if (*a*) the a_{ij}'s (coefficients of the structural variables in the constraints) are restricted to the values 0 and/or 1 and (*b*) There exists homogeneity of units among the constraints.

Approach to Solution to A Transportation Problem By Using Transportation Algorithm

The steps used in getting a solution to a transportation problem is given below:

Initial Basic Feasible Solution

Step 1. Balancing the given problem. Balancing means check whether sum of availability constraints must be equals to sum of requirement constraints. That is $\sum b_i = \sum d_j$. Once they are equal, go to step two. If not by opening a *Dummy row* or *Dummy column* balance the problem. The *cost coefficients of dummy cells are zero*. If $\sum b_i$ is greater than $\sum d_j$, then open a dummy column, whose requirement constraint is equals to $\sum b_i - \sum d_j$ and the cost coefficient of the cells are zeros. In case if $\sum d_j$ is greater than $\sum b_i$, then open a dummy row, whose availability constraint will be equals to $\sum d_j$ b_i and the cost coefficient of the cells are zeros. Once the balancing is over, then go to second step. Remember while solving general linear programming problem to convert an inequality into an equation, we add (for maximization problem) a slack variable. In transportation problem, the dummy row or dummy column, exactly similar to a slack variable.

Step II. A .Basic feasible solution can be obtained by three methods, they are

(a) North - west corner method.

(b) Least - cost cell method. (Or Inspection method Or Matrix minimum - row minimum
- column minimum method)

(c) Vogel's Approximation Method, generally known as VAM.

After getting the basic feasible solution (*b.f.s.*) give **optimality test** to check whether the solution is optimal or not.

There are two methods of giving optimality test:

(a) Stepping Stone Method.

(b) Modified Distribution Method, generally known as **MODI** method.

Properties of a Basic feasible Solution

1. The allocation made must satisfy the rim requirements, *i.e.*, it must satisfy availability constraints and requirement constraints.
2. It should satisfy non negativity constraint.
3. Total number of allocations must be equal to $(m + n - 1)$, where 'm' is the number of rows and 'n' is the number of columns. Consider a value of $m = 4$ and $n = 3$, *i.e.* 4×3 matrix. This will have four constraints of $\leq$ type and three constraints of $\geq$ type. Totally it will have $4 + 3$ (*i.e* $m + n$) inequalities. If we consider them as equations, for solution purpose, we will have 7 equations. In case, if we use simplex method to solve the problem, only six rather than seen structural constraints need to be specified. In view of the fact that the sum of the origin capacities (availability constraint) equals to the destination requirements (requirement constraint) *i.e.*, $\square b_i = \square d_j$, any solution satisfying six of the seven constraints will

automatically satisfy the last constraint. In general, therefore, if there are 'm' rows and 'n' columns, in a given transportation problem, we can state the problem completely with $m + n - 1$ equations. This means that one of the rows of the simplex tableau represents a redundant constraint and, hence, can be deleted. This also means that a basic feasible solution of a transportation problem has only $m + n - 1$ positive components. If $\sum b_i = \sum d_j$, it is always possible to get a basic feasible solution by North-west corner method, Least Cost cell method or by VAM.

Basic Feasible Solution by North - West corner Method

Let us take a numerical example and discuss the process of getting basic feasible solution by various methods.

Example 1. Four factories, *A*, *B*, *C* and *D* produce sugar and the capacity of each factory is given below: Factory *A* produces 10 tons of sugar and *B* produces 8 tons of sugar, *C* produces 5 tons of sugar and that of *D* is 6 tons of sugar. The sugar has demand in three markets *X*, *Y* and *Z*. The demand of market *X* is 7 tons, that of market *Y* is 12 tons and the demand of market *Z* is 4 tons. The following matrix gives the transportation cost of 1 ton of sugar from each factory to the destinations. Find the Optimal Solution for least cost transportation cost.

Factories.	*Cost in Rs. per ton (× 100) Markets.*			*Availability in tons.*
	X	*Y*	*Z*	
A	4	3	2	10
B	5	6	1	8
C	6	4	3	5
D	3	5	4	6
Requirement in tons.	7	12	4	$\sum b = 29$, $\sum d = 23$

Here b is greater than d hence we have to open a dummy column whose requirement constraint is 6, so that total of availability will be equal to the total demand. Now let get the basic feasible solution by three different methods and see the advantages and disadvantages of these methods. After this let us give optimality test for the obtained basic feasible solutions.

a) North- west corner method

(i) Balance the problem. That is see whether $\sum b_i = \sum d_j$. If not open a dummy column or dummy row as the case may be and balance the problem.

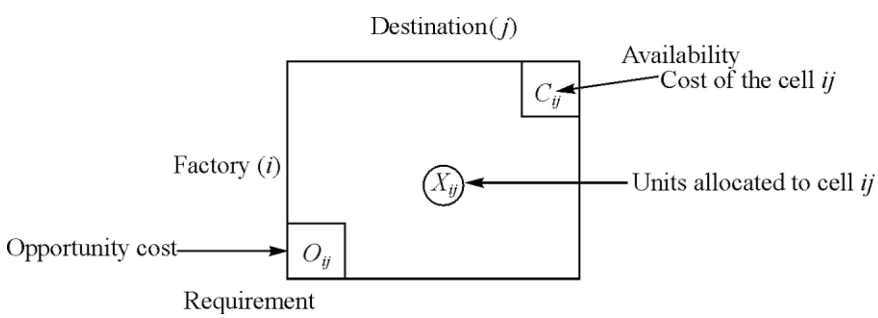
(ii) Start from the left hand side top corner or cell and make allocations depending on the availability and requirement constraint. If the availability constraint is less than the requirement constraint, then for that cell make allocation in units which is equal to the availability constraint. In general, verify which is the smallest among the availability and requirement and allocate the smallest one to the cell under question. Then proceed allocating either sidewise or down- ward to satisfy the rim requirement. Continue this until all the allocations are over.

(iii) Once all the allocations are over, *i.e.*, both rim requirement (column and row *i.e*., availability and requirement constraints) are satisfied, write allocations and calculate the cost of transportation.

Solution by North-west corner method:

	X	Y	Z	Dummy	Availability
A	4 (7)	3 (3)	2	0	10
B	5	6 (8)	1	0	8
C	6	4 (1)	3 (4)	0	5
D	3	5	4 (1)	0 (5)	6
Requirement.	7	12	5	5	29

For cell *AX* the availability constraint is 10 and the requirement constraint is 7. Hence 7 is smaller than 10, allocate 7 to cell *AX*. Next 10 – 7 = 3, this is allocated to cell *AY* to satisfy availability requirement. Proceed in the same way to complete the allocations. Then count the allocations, if it is equals to $m + n - 1$, then the solution is ***basic feasible solution***. The solution, we got have **7** allocations which is = 4 + 4 – 1 = 7. Hence the solution is basic feasible solution.



Now allocations are:

From	To	Units in tons	Cost in Rs.
A	*X*	7	7 × 4 = 28
A	*Y*	3	3 × 3 = 09
B	*Y*	8	8 × 6 = 48
C	*Y*	1	1 × 4 = 04
C	*Z*	4	4 × 3 = 12
D	*Z*	1	1 × 4 = 04
D	DUMMY	5	5 × 0 = 00
	Total in Rs.		105

Solution by Least cost cell (or inspection) Method: (Matrix Minimum method)

(*i*) Identify the lowest cost cell in the given matrix. In this particular example it is = 0. Four cells of dummy column are having zero. When more than one cell has the same cost, then both the cells are competing for allocation. This situation in transportation problem is known as **tie**. To break the tie, select any one cell of your choice for allocation. Make allocations to this cell either to satisfy availability constraint or requirement constraint. Once one of these is satisfied, then mark crosses (×) in all the cells in the row or column which ever has completely allocated. Next search for lowest cost cell. In the given problem it is cell BZ which is having cost of Re.1/- Make allocations for this cell in similar manner and mark crosses to the cells in row or column which has allocated completely. Proceed this way until all allocations are made. Then write allocations and find the cost of transportation. As the total number of allocations are **7** which is equals to 4 + 4 – 1 = 7, the solution is basic feasible solution.

	X	Y	Z	Dummy	Availability
A	× [4]	8 [3]	2 [2]	× [0]	10(8) (0) iv
B	× [5]	× [6]	3 [1]	5 [0]	8(3) (0) ii
C	1 [6]	4 [4]	× [3]	× [0]	5(4) (0) viii
D	6 [3]	× [5]	× [4]	× [0]	6 (0) v
Requirement.	7 (1) (0) vi	12 (4) (0) vii	5 (2) (0) iii	5 (0) i	29

(**Note:** The numbers under and side of rim requirements shows the sequence of allocation and the units remaining after allocation)

Allocations are:

From	*To*	*Units in tons*	*Cost in Rs.*
A	Y	8	8 × 3 = 24
A	*Z*	2	2 × 2 = 04
B	*Z*	3	3 × 1 = 03
B	DUMMY	5	5 × 0 = 00
C	*X*	1	1 × 6 = 06
C	*Y*	4	4 × 4 = 16
D	*X*	6	6 × 3 = 18
		Total in Rs.	**71**

Solution by Vogel's Approximation Method: (Opportunity cost method)

(i) In this method, we use concept of opportunity cost. Opportunity cost is the penalty for not taking correct decision. To find the row opportunity cost in the given matrix deduct the smallest element in the row from the next highest element. Similarly to calculate the column opportunity cost, deduct smallest element in the column from the next highest element. Write row opportunity costs of each row just by the side of availability constraint and similarly write the column opportunity cost of each column just below the requirement constraints. These are known as penalty column and penalty row.

The rationale in deducting the smallest element form the next highest element is:

Let us say the smallest element is 3 and the next highest element is 6. If we transport one unit

through the cell having cost Rs.3/-, the cost of transportation per unit will be Rs. 3/-. Instead we transport through the cell having cost of Rs.6/-, then the cost of transportation will be Rs.6/- per unit. That is for not taking correct decision; we are spending Rs.3/- more (Rs.6 – Rs.3 = Rs.3/-). This is the penalty for not taking correct decision and hence the opportunity cost. This is the lowest opportunity cost in that particular row or column as we are deducting the smallest element form the next highest element.

Note: If the smallest element is three and the row or column having one more three, then we have to take next highest element as three and not any other element. Then the opportunity cost will be zero. In general, if the row has two elements of the same magnitude as the smallest element then the opportunity cost of that row or column is zero.

(ii) Write row opportunity costs and column opportunity costs as described above.

(iii) Identify the highest opportunity cost among all the opportunity costs and write a tick (√) mark at that element.

(iv) If there are two or more of the opportunity costs which of same magnitude, then select any one of them, to break the tie. While doing so, see that both availability constraint and requirement constraint are simultaneously satisfied. If this happens, we may not get basic feasible solution *i.e* solution with $m + n - 1$ allocations. As far as possible see that both are not satisfied simultaneously. In case if inevitable, proceed with allocations. We may not get a solution with, $m + n - 1$ allocations. For this we can allocate a small element epsilon (∈) to any one of the empty cells. This situation in transportation problem is known as degeneracy. (This will be discussed once again when we discuss about optimal solution).

In transportation matrix, all the cells, which have allocation, are known as **loaded cells** and those, which have no allocation, are known as **empty cells**.

(Note: All the allocations shown in matrix 1 to 6 are tabulated in the matrix given below:**)**

(1)

	X	Y	Z	DMY		
A	4	3	2	0	10	(2)
B	5	6	1	0	8	(1)
C	6	4	3	0	5	**(3)**
D	3	5	4	0 **5**	6	**(3)** ←
	7	12	5	5	29	
	(1)	(1)	(1)	(0)		

(4)

	X	Y		
A	4	3	10	(1)
B	5	6	3	(1)
C	~~6~~	~~4~~ **5**	5	**(2)** ←
	6	12	18	
	(1)	(1)		

(2)

	X	Y	Z	
A	4	3	2	10
B	5	6	1 **5**	8
C	6	4	3	5
D	3	5	4	1
	5	12	5	24
	(1)	(1)	(1)	

(5)

	X	Y		
A	4	**7** 3	10(1)	
B	5	6	3(1)	
	6	7	13	
	(1)	**(3)** ↑		

(3)

	X	Y		
A	4	3	10	(1)
B	5	6	3	(1)
C	6	4	5	**(2)**
D	**1** 3	5	1	**(2)** ←
	7	12	19	
	(1)	(1)		

(6)

		×	
(1)	A	4 **3**	3
(4) ←			
(1)	B	**3** 5	3
(1)		6	6

Consider matrix (1), showing cost of transportation and availability and requirement constraints. In the first row of the matrix, the lowest cost element is 0, for the cell A-Dummy and next highest element is 2, for the cell AZ. The difference is 2 – 0 = 2. The meaning of this is, if we transport the load through the cell A-Dummy, whose cost element is 0, the cost of transportation will be = Rs.0/- for each unit transported. Instead, if we transport the load through the cell, AZ whose cost element is Rs. 2/- the transportation cost is = Rs.2/- for each unit we transport. This means to say if we take decision to send the goods through the cell AZ, whose cost element is Rs.2/- then the management is going to loose Rs. 2/- for every unit it transport through *AZ*. Suppose, if the management decide to send load through the cell *AX*, Whose cost element is Rs.4/-, then the penalty or the opportunity cost is *Rs*.4/-. We write the minimum opportunity cost of the row outside the matrix. Here it is shown in brackets. Similarly, we find the column opportunity costs for each column and write at the bottom of each corresponding row (in brackets). After writing all the opportunity costs, then we select the highest among them. In the given matrix it is Rs.3/- for the rows *D* and *C*. **This situation is known as tie.**

When tie exists, select any of the rows of your choice. At present, let us select the row *D*. Now in thatrow select the lowest cost cell for allocation. This is because; our objective is to minimize the transportation cost. For the problem, it is *D*-dummy, whose cost is zero. For this cell examine what is available and what is required? Availability is 6 tons and requirement is 5 tons. Hence allocate 5 tons to this cell and cancel the dummy row from the problem. Now the matrix is reduced to 3 × 4. Continue the above procedure and for every allocation the matrix goes on reducing, finally we get all allocations are over. Once the allocations are over, count them, if there are $m + n - 1$ allocations, then the solution is basic feasible solution. Otherwise, the **degeneracy** occurs in the problem. To solve degeneracy, we have to **add epsilon** (∈), a small element to one of the empty cells. This we shall discuss, when we come to discuss optimal solution. Now for the problem the allocations are:

From	*To*	*Load*	*Cost in Rs.*
A	*X*	3	3 × 4 = 12
A	*Y*	7	7 × 3 = 21
B	*X*	3	3 × 5 = 15
B	*Z*	5	5 × 1 = 05
C	*Y*	5	5 × 4 = 20
D	*X*	1	1 × 3 = 03
D	DUMMY	5	5 × 0 = 00
		Total Rs.	**76**

Now let us compare the three methods of getting basic feasible solution:

North – west corner method.	**Inspection or least cost cell method**	**Vogel's Approximation Method.**
1. The allocation is made from the left hand side top corner irrespective of the cost of the cell.	The allocations are made depending on the cost of the cell. Lowest cost is first selected and then next highest etc.	The allocations are made depending on the opportunity cost of the cell.

2. As no consideration is given to the cost of the cell, naturally the total transportation cost will be higher than the other methods.	As the cost of the cell is considered while making allocations, the total cost of transportation will be comparatively less.	As the allocations are made depending on the opportunity cost of the cell, the basic feasible solution obtained will be very nearer to optimal solution.
3. It takes less time. This method is suitable to get basic feasible solution quickly.	The basic feasible solution, we get will be very nearer to optimal solution. It takes more time than northwest coronermethod.	It takes more time for getting basic Feasible solution. But the solution we get will be very nearer to Optimal solution.
4. When basic feasible solution alone is asked, it is better to go for northwest corner method.	When optimal solution is asked, better to go for inspection method for basic feasible solution and MODI for optimal solution.	VAM and MODI is the best option to get optimal solution.

In the problem given, the total cost of transportation for Northwest corner method is Rs. 101/-. The total cost of transportation for Inspection method is Rs. 71/- and that of VAM is Rs. 76/-. The total cost got by inspection method appears to be less. That of Northwest coroner method is highest. The cost got by VAM is in between.

Now let us discuss the method of getting optimal solution or methods of giving optimality test for basic feasible solution.

Optimality Test: (Approach to Optimal Solution)

Once, we get the basic feasible solution for a transportation problem, the next duty is to test whether the solution got is an optimal one or not? This can be done by two methods. (*a*) By Stepping Stone Method, and (*b*) By Modified Distribution Method, or MODI method.

(a) Stepping stone method of optimality test

To give an optimality test to the solution obtained, we have to find the opportunity cost of empty cells. As the transportation problem involves decision making under certainty, we know that an optimal solution must not incur any positive opportunity cost. Thus, we have to determine whether any positive opportunity cost is associated with a given progarmme, *i.e.*, for empty cells. **Once the opportunity cost of all empty cells are negative, the solution is said to be optimal.** In case any one cell has got positive opportunity cost, then the solution is to be modified. The Stepping stone method is used for finding the opportunity costs of empty cells. Every empty cell is to be evaluated for its opportunity cost. To do this the methodology is:

1. Put a small ‘+’ mark in the empty cell.
2. Starting from that cell draw a loop moving horizontally and vertically from loaded cell to loaded cell. Remember, there should not be any diagonal movement. We have to take turn only at loaded cells and move to vertically

downward or upward or horizontally to reach another loaded cell. In between, if we have a loaded cell, where we cannot take a turn, ignore that and proceed to next loaded cell in that row or column.

3. After completing the loop, mark minus (–) and plus (+) signs alternatively.
4. Identify the lowest load in the cells marked with negative sign.
5. This number is to be added to the cells where plus sign is marked and subtract from the load of the cell where negative sign is marked.
6. Do not alter the loaded cells, which are not in the loop.
7. The process of adding and subtracting at each turn or corner is necessary to see that rim requirements are satisfied.
8. Construct a table of empty cells and work out the cost change for a shift of load from loaded cell to loaded cell.
9. If the cost change is positive, it means that if we include the evaluated cell in the programme, the cost will increase. If the cost change is negative, the total cost will decrease, by including the evaluated cell in the programme.
10. The negative of cost change is the opportunity cost. Hence, in the optimal solution of transportation problem empty cells should not have positive opportunity cost.
11. Once all the empty cells have negative opportunity cost, the solution is said to be optimal.

One of the drawbacks of stepping stone method is that we have to write a loop for every empty cell. Hence it is tedious and time consuming. Hence, for optimality test we use MODI method rather than the stepping stone method.

Let us take the basic feasible solution we got by Vogel's Approximation method and give optimalitytest to it by stepping stone method.

Basic Feasible Solution obtained by VAM:

	X	Y	Z	Dummy	Availability
A	4 3 –	3 7 +	2 –2	0 +1	10
B	5 3	6 –2	1 5	0 +2	8
C	6 –1	4 5 –	3 –1	0 + 0	5
D	3 1 + 3	5 –3 5	4 –5 4	0 5 – 0	6
Requirement.	7	12	5	5	29

Table showing the cost change and opportunity costs of empty cells:

Table I.

S.No.	*Empty Cell*	*Evalution Loop formation*	*Cost change in Rs.*	*Opportunity cost -(Cost change)*
1.	AZ	+AZ – AX + BX – BZ	+2 – 4 + 5 – 1 = +2	–2
2	A Dummy	+ A DUMMY – AX + BX – B DUMMY	+0 – 4 + 3 – 0 = –1	+1
3	BY	+ BY – AY + AX – BX	+6 – 3 + 4 – 5 = +2	-2
4	B DUMMY	+ B DUMMY – BX + DX – D DUMMY	+0 – 5 +3 – 0 = –2	+2
5	CX	+CX – CY + AX – AY	6 – 4 + 3 – 4 = +1	– 1
6	CZ	+CZ – BZ + BX –AX + AY – CY	+2 –1 +5 – 4 +5 – 4 =+1	– 1
7	C DUMMY	+ C DUMMY – D DUMMY + DX –AX + AY – CY	+ 0 – 0 +3 – 4 +3 – 4 = –2	+2
8	DY	+DY – DX + AX – AY	+5 – 3 +4 – 3 = +3	– 3
9	DZ	+DZ – DX +BX – BZ	+4 – 3 + 5 – 1 = +5	– 5

In the table 1 cells A DUMMY, B DUMMY, C DUMMY are the cells which are having positive opportunity cost. Between these two cells B DUMMY and C DUMMY are the cells, which are having higher opportunity cost i.e Rs. 2/ - each. Let us select any one of them to include in the improvement of the present programme. Let us select C DUMMY.

	X	Y	Z	Dummy	Availability
A	4 –8	3 10	2 –4	0 +1	10
B	5 3 –	6 0	1 5	0 +2 +	8
C	6 –3	4 2	3 –3	0 3	5
D	3 4 +	5 –1	4 –5	0 2	6
Requirement.	7	12	5	5	29

Table II.

S.No.	Empty Cell	Evalution Loop formation	Cost change in Rs.	Opportunity Cost
1	AX	+AX –DX + D DUMMY – C DUMMY + CY – AY	+ 4 – 3 + 0 – 0 + 4 – 3 = + 2	–2
2	AX	AZ – AY + CY – C DUMMY + D DUMMY – DX+ BX – BZ	+ 2 – 3 + 4 – 0 + 0 – 3 + 3 – 0 = + 4	–4
3	ADUMMY	+ A DUMMY – AY + DX –D DUMMY	+ 0 – 4 + 3 – 0 = – 1	+1
4	BY	+BY – BX + DX – D DUMMY +C DUMMY – CY	+ 6 –5 + 3 –0 + 0 – 4 = 0	0
5	B DUMMY	+ B DUMMY – BX + DX – D DUMMY	+ 0 – 5 + 3 – 0 = –2	+2
6	CX	+ CX – DX + D DUMMY – C DUMMY	+ 6 –3 + 0 – 0 = +3	–3
7	CZ	+ CZ – C DUMMY + D DUMMY – DX + BX – BZ	+ 2 – 0 + 0 – 3 + 5 – 1 = + 3	–3
8	DY	DY – CY + C DUMMY – D DUMMY	+ 5 – 4 + 0 – 0 = 1	–1
9	DZ	+ DZ – DX + BX – BZ	+ 4 – 3 +5 – 1 = + 5	–5

Cells A DUMMY and B DUMMY are having positive opportunity costs. The cell B DUMMY is having higher opportunity cost. Hence let us include this cell in the next programme to improve the solution.

Table III.

S.No.	Empty Cell	Evaluation Loop formation	Cost change in Rs.	Opportunity Cost
1	AX	+AX – AY + CY – C DUMMY + B DUMMY – BX	+4 – 3 + 4 – 0 + 0 – 5 = 0	0
2	AZ	+ AZ – BZ + B DUMMY – C DUMMY + CX – AX	+2 –1 + 0 – 0 + 4 – 3 =+ 2	–2

3	A DUMMY	+ A DUMMY – C DUMMY + CY – AY	+0 – 0 + 4 – 3 = +1	–1
4	BY	+ BY – B DUMMY + C DUMMY – CY	+6 – 0 + 0 – 4 = + 2	–2
5	CX	+ CX – BX + B DUMMY – C DUMMY	+6 – 5 + 0 – 0 = +1	–1
6	CZ	+ CZ – BZ + B DUMMY – C DUMMY	+2 – 1 + 0 – 0 = +1	–1
7	DY	+DY – CY + C DUMMY – B DUMMY + BX – DX	+5 – 4 + 0 – 0 + 5 – 3 = +3	–3
8	DZ	+ DZ – BZ + BX – DX	+4 – 1 + 5 – 3 = +5	–5
9	D DUMMY	+ D DUMMY – DX + BX – B DUMMY	+ 0 – 3 + 5 – 0 = +2	–2

All the empty cells have negative opportunity cost hence the solution is optimal. The **allocations are:**

S.No	*Loaded cell*	*Load*	*Cost in Rs.*
1	AY	10	10 × 3 = 30
2	BX	01	01 × 5 = 05
3	BZ	05	05 × 1 = 05
4	B DUMMY	02	02 × 0 = 00
5	CY	02	02 × 4 = 08
6	C DUMMY	03	03 × 0 = 00
7	DX	06	06 × 3 = 18
	Total in Rs.		66

Total minimum transportation cost is Rs. 66/-

Optimal allocation.

	X	Y	Z	Dummy	Availability
A	4 0	3 10	2 0	0 –1	10
B	5 1	6 –2	1 5	0 2	8
C	6 –1	4 2	3 –1	0 3	5
D	3 6	5 –2	4 –5	0 –2	6
Requirement.	7	12	5	5	29

Modified Distribution Method of Optimality test

In stepping stone method, we have seen that to get the opportunity cost of empty cells, for every cell we have to write a loop and evaluate the cell, which is a laborious process. In MODI (Modified DIstribution method, we can get the opportunity costs of empty cells without writing the loop. After getting the opportunity cost of all the cells, we have to select the cell with highest positive opportunity cost for including it in the modified solution.

Steps in MODI method:

1. Select row element (u_i) and Column element (v_j) for each row and column, such that $u_i + v_j$ = the actual cost of loaded cell. In MODI method we can evaluate empty cells simultaneously and get the opportunity cost of the cell by using the formula $(u_i + v_j) – C_{ij}$, where C_{ij} is the actual cost of the cell.
2. In resource allocation problem (maximization or minimization method), we have seen that once any variable becomes basis variable, *i.e.*, the variable enters the programme; its opportunity cost or net evaluation will be zero. Here, in transportation problem also, once any cell is loaded, its opportunity cost will be zero. Now the opportunity cost is given by $(u_i + v_j) – C_{ij}$, which is, equals to zero for a loaded cell. *i.e.* $(u_i + v_j) – C_{ij} = 0$ which means, $(u_i + v_j) = C_{ij}$. Here $(u_i + v_j)$ is known as **implied cost of the cell. For any loaded cell the implied cost is equals to actual cost of the cell as its opportunity cost is zero. For any empty cell, (implied cost – actual cost) will give opportunity cost.**
3. How to select u_i and v_j? The answer is:

(a) Write arbitrarily any one of them against a row or against a column. The written u_i or vj may be any whole number *i.e* u_i or v_j may be ≤ or ≥ to zero. By using the formula $(u_i + v_j) = C_{ij}$ for a loaded cell, we can write the other row or column element. For example, if the actual cost of the cell $C_{ij} = 5$ and arbitrarily we have selected $u_i = 0$, then v_j is given by $u_i + v_j = 0 + v_j = 5$. Hence $v_j = –5$. Like this, we can go from loaded cell to loaded cell and complete entering of all u_i *s* and v_j *s*.

(b) Once we get all u_i *s* and v_j *s*, we can evaluate empty cells by using the formula $(u_i + v_j)$ – Actual cost of the cell = opportunity cost of the cell, and write the opportunity cost of each empty cell at left hand bottom corner.

(c) Once the opportunity costs of all empty cells are negative, the solution is said to be optimal. In case any cell is having the positive opportunity cost, the programme is to be modified.

Remember the formula that IMPLIED COST OF A CELL = $u_i + v_j$

Opportunity cost of loaded cell is zero *i.e* $(u_i + v_j)$ = Actual cost of the cell.

Opportunity cost of an empty cell = implied cost – actual cost of the cell = $(u_i + v_j) – C_{ij}$

***(d)* In case of degeneracy, *i.e.* in a basic feasible solution, if the number of loaded cells are not equals to $m + n – 1$, then we have to add a small element *epsilon* (∈), to any empty cell to make the number of loaded cells equals to $m + n – 1$. While adding '∈' we must be careful enough to see that this ∈ should not form a closed loop when we draw horizontal and vertical lines from loaded cell to loaded cell. In case the cell to which we have added ∈ forms a closed**

Implied cost	Actual cost	Action
$u_i + v_j >$	C_{ij}	A better programme can be designed by including this cell in the solution.
$u_i + v_j =$	C_{ij}	Indifferent; however, an alternative programme with same total cost can be written by including this cell in the programme.
$u_i + v_j <$	C_{ij}	Do not include this cell in the programme.

Now let us take the basic feasible solution obtained by VAM method and apply MODI method of optimality test.

	X	Y	Z	Dummy	Availability	Row No. u_i
A	4 (3) −	3 (7) +	2	0	10	4
B	5 (3)	6	1 (5)	0	8	5
C	6	4 (5)	3	0 +	5	5
D	3 (1) +	5	4	0 (5) −	6	3
Requirement.	7	12	5	5	29	
Column No v_j	0	–1	–4	–3		

Basic feasible solution got by VAM method

The cell C DUMMY is having a positive opportunity cost. Hence we have to include this cell in the programme. The solution has $m + n – 1$ allocations.

The cell B DUMMY is having a positive opportunity cost. Thïs is to be included in the modified programme.

	X	Y	Z	DUMMY	Availability	Row No. u_i
A	$\boxed{4}$ 0	$\boxed{3}$ 10	$\boxed{2}$ –2	$\boxed{0}$ –1	10	–1
B	$\boxed{5}$ 1	$\boxed{6}$ –2	$\boxed{1}$ 5	$\boxed{0}$ 2	8	0
C	$\boxed{6}$ –1	$\boxed{4}$ 2	$\boxed{3}$ –2	$\boxed{0}$ 3	5	0
D	$\boxed{3}$ 6	$\boxed{5}$ –1	$\boxed{4}$ –3	$\boxed{0}$ 0	6	0
Requirement .	7	12	5	5	29	
Column Number: v_j	5	4	1	0		

As the opportunity cost of all empty cells are negative, the solution is optimal. The solution has $m+ n – 1$ allocations.

The allocations are:

S.No	*Loaded Cell*	*Load*	*Cost in Rs.*
1	AY	10	10 × 3 = 30
2.	BX	01	01 × 5 = 05
3.	BZ	05	05 × 1 = 05
4.	B DUMMY	02	02 × 0 = 00
5.	CY	02	02 × 4 = 08
6.	C DUMMY	03	03 × 0 = 00
7.	CX	06	06 × 3 = 18
	Total Cost in Rs.		66

Readers can verify the optimal solution got by Stepping stone method and the MODI method they are same. And they can also verify the opportunity costs of empty cells they are also same. This is the advantage of using MODI method to give optimality test. Hence the combination of VAM and MODI can be conveniently used to solve the transportation problem when optimal solution is asked.

UNIT – III ASSIGNMENT MODEL

INTRODUCTION

In earlier discussion in chapter 1 and 2, we have dealt with two types of linear programming problems, *i.e.* Resource allocation method and Transportation model. We have seen that though we can use simplex method for solving transportation model, we go for transportation algorithm for simplicity. We have also discussed that how a resource allocation model differ from transportation model and similarities between them. Now we have another model comes under the class of linear programming model, which looks alike with transportation model with an objective function of minimizing the time or cost of manufacturing the products by allocating one job to one machine or one machine to one job or one destination to one origin or one origin to one destination only. This type of problem is given thename **ASSIGNMENT MODEL**. Basically assignment model is a minimization model. If we want to maximize the objective function, then there are two methods. One is to subtract all the elements of the matrix from the highest element in the matrix or to multiply the entire matrix by –1 and continue with the procedure. For solving the assignment problem we use Assignment technique or Hungarian method or Flood's technique. All are one and the same. Above, it is mentioned that one origin is to be assignedto one destination. This feature implies the existence of two specific characteristics in linear programming problems, which when present, give rise to an assignment problem. The first one being **the pay of matrix for a given problem is a square matrix** and the second is the optimum solution (or any solution with given constraints) for the problem is such that there can be **one and only one assignment in a given row or column of the given payoff matrix.** The transportation model is a special case of linear programming model (Resource allocation model) and assignment problem is a special case of transportation model, therefore it is also a special case of linear programming model. Hence it must have all the properties of linear programming model. That is it must have: (*i*) an objective function, (*ii*) it must have structural constraints, (*iii*) It must have non-negativity constraint and (*iv*) The relationship between variables and constraints must have linear relationship. In our future discussion, we will see that the assignment problem has all the above properties.

The Problem

There are some types in assignment problem. They are:

(i) Assigning the jobs to machines when the problem has square matrix to minimize the time required to complete the jobs. Here the number of rows *i.e.* jobs are equals to the number of machines *i.e.* columns. The procedure of solving will be discussed in detail in this section.

(ii) The second type is maximization type of assignment problem. Here we have to assign certain jobs to certain facilities to maximize the returns or maximise the effectiveness. This is also discussed in problem number 5.2.

(iii) Assignment problem having non-square matrix. Here by adding a dummy row or dummy columns as the case may be, we can convert a non-square matrix into a square matrix and proceed further to solve the problem. This is done in problem number.5.9.

(iv) Assignment problem with restrictions. Here restrictions such as a job cannot be done on a certain machine or a job cannot be allocated to a certain facility may be specified. In such cases, we should neglect such cell or give a high penalty to that cell to avoid that cell to enter into the programme.

(v) Traveling sales man problem (cyclic type). Here a salesman must tour certain cities starting from his hometown and come back to his hometown after visiting all cities. This type of problem can be solved by Assignment technique and is solved in problem 5.14.

Let us take that there are 4 jobs, *W*, *X*, *Y* and *Z* which are to be assigned to four machines, *A*, *B*, *C* and *D*. Here all the jobs have got capacities to machine all the jobs. Say for example that the job *W* is to drill a half and inch hole in a Wooden plank, Job *X* is to drill one inch hole in an Aluminum plate and Job *Y* is to drill half an inch hole in a Steel plate and job *Z* is to drill half an inch hole in a Brass plate. The machine *A* is a Pillar type of drilling machine, the machine *B* is Bench type of drilling machine, Machine *C* is radial drilling machine and machine *D* is an automatic drilling machine. This gives an understanding that all machines can do all the jobs or all jobs can be done on any machine. The cost ortime of doing the job on a particular machine will differ from that of another machine, because of overhead expenses and machining and tooling charges. The objective is to minimize the time or cost of manufacturing all the jobs by allocating one job to one machine. Because of this character, *i.e.* one to one allocation, the assignment matrix is always a square matrix. If it is not a square matrix, then the problem is unbalanced. Balance the problem, by opening a dummy row or dummy column with its cost or time coefficients as zero. Once the matrix is square, we can use assignment algorithm or Flood's technique or Hungarian method to solve the problem.

Jobs	Machines (Time in hours)				Availability
	A	*B*	*C*	*D*	
W	C_{11}	C_{12}	C_{13}	C_{14}	1
X	C_{21}	C_{22}	C_{23}	C_{24}	1
Y	C_{31}	C_{32}	C_{33}	C_{34}	1
Z	C_{41}	C_{42}	C_{43}	C_{44}	1
Requirement:	1	1	1	1	

Mathematical Model:

Minimize Z = $\sum \sum C_{ij} \, x_{ij} \rightarrow$ **Objective Constraint.**

For *i* and *j* = 1 to *n*

(Each machine to one job only) (Each job to one machine only)

And

X_{ij} = 0 for all values of *j* and *i*. Non-negativity constraint.

Comparison between Transportation Problem and AssignmentProblem

Now let us see what are the similarities and differences between Transportation problem and Assignment Problem.

Similarities

1. Both are special types of linear programming problems.
2. Both have objective function, structural constraints, and non-negativity constraints. And the relationship between variables and constraints are linear.
3. The coefficients of variables in the solution will be either 1 or zero in both cases.
4. Both are basically minimization problems. For converting them into maximization problem same procedure is used.

Differences

Transportation Problem	*Assignment Problem.*
1. The problem may have rectangular matrixor square matrix. 2.The rows and columns may have any number of allocations depending on the rimconditions. 3.The basic feasible solution is obtained by northwest corner method or matrix minimummethod or VAM 4.The optimality test is given by stepping stone method or by MODI method. 5.The basic feasible solution must have m + n – 1 allocations. 6.The rim requirement may have any numbers (positive numbers). 7.In transportation problem, the problem deals with one commodity being moved from various origins to various destinations.	1.The matrix of the problem must be a square matrix. 2.The rows and columns must have one to one allocation. Because of this property, the matrix mustbe a square matrix. 3.The basic feasible solution is obtained by Hungarian method or Flood's technique or by Assignment algorithm. 4.Optimality test is given by drawing minimum number of horizontal and vertical lines to cover all the zeros in the matrix. 5.Every column and row must have at least one zero. And one machine is assigned to one job and vice versa. 6. The rim requirements are always 1 each for every row and one each for every column. 7.Here row represents jobs or machines and columnsrepresents machines or jobs.

APPROACH TO SOLUTION

Let us consider a simple example and try to understand the approach to solution and then discuss complicated problems.

1. Solution by visual method

In this method, first allocation is made to the cell having lowest element. (In case of maximization method, first allocation is made to the cell having highest element). If there is more than one cell having smallest element, tie exists and allocation may be made to any one of them first and then second one is selected. In such cases, there is a possibility of getting alternate solution to the problem. This method is suitable for a matrix of size 3 × 4 or 4 × 4. More than that, we may face difficulty in allocating.

Problem 1.

There are 3 jobs *A*, *B*, and *C* and three machines *X*, *Y*, and *Z*. All the jobs can be processed on all machines. The time required for processing job on a machine is given below in the form of matrix. Make allocation to minimize the total processing time.

Machines (time in hours)

Jobs	*X*	*Y*	*Z*
A	11	16	21
B	20	13	17
C	13	15	12

Allocation: *A* to *X*, *B* to *Y* and *C* to *Z* and the total time = 11 + 13 +12 = 36 hours. (Since 11 is least, Allocate *A* to *X*, 12 is the next least, Allocate *C* to *Z*)

2. Solving the assignment problem by enumeration

Let us take the same problem and workout the solution.

Machines (time in hours)

C	13	15	12
Jobs	X	Y	Z
A	11	16	21
B	20	13	17
C	13	15	12

S.No	Assignment	Total cost in Rs.
1	AX BY CZ	11 + 13 + 12 = 36
2	AX BZ CY	11 + 17 + 15 = 43
3	AY BX CZ	16 + 20 + 12 = 48

4	AY BZ CX	16 + 17 + 13 = 46
5	AZ BY CX	21 + 13 + 13 = 47
6	AZ BX CY	21 + 20 + 15 = 56

Like this we have to write all allocations and calculate the cost and select the lowest one. If more than one assignment has same lowest cost then the problem has alternate solutions.

3. Solution by Transportation method

Let us take the same example and get the solution and see the difference between transportation problem and assignment problem. The rim requirements are 1 each because of one to one allocation.

Machines (Time in hours)

Jobs	*X*	*Y*	*Z*	*Available*
A	11	16	21	1
B	20	13	17	1
C	13	15	12	1
Req	1	1	1	3

By using northwest corner method the assignments are:

Machines (Time in hours)

Jobs	*X*	*Y*	*Z*	*Available*
A	1	E		1
B		1	ϵ	1
C			1	1
Req	1	1	1	3

As the basic feasible solution must have $m + n - 1$ allocations, we have to add 2 epsilons. Next we have to apply optimality test by MODI to get the optimal answer.

This is a time consuming method. Hence it is better to go for assignment algorithm to get the solution for an assignment problem.

4. Hungarian Method / Flood's technique / Assignment algorithm: (opportunity cost method)

Let us once again take the same example to workout with assignment algorithm.

Machines (time in hours)

Jobs	*X*	*Y*	*Z*
A	11	16	21

B	20	13	17
C	13	15	12

Step 1. Deduct the smallest element in each row from the other elements of the row. The matrix thus got is known as **Row opportunity cost matrix (ROCM). The logic here is if we assign the job to any machine having higher cost or time, then we have to bear the penalty. If we subtract smallest element in the row or from all other element of the row, there will be at least one cell having zero, i.e zero opportunity cost or zero penalty. Hence that cell is more competent one for assignment.**

Step 2. Deduct the smallest element in each column from other elements of the column. The matrix thus got is known as **Column opportunity cost matrix (COCM). Here also by creating a zero by subtracting smallest element from all other elements we can see the penalty that one has to bear. Zero opportunity cell is more competent for assignment.**

Step 3. Add COCM and ROCM to get the **Total opportunity cost matrix (TOCM).**

Step 4. (modified): Total opportunity cost matrix can be got by simplify doing row operation on Column opportunity matrix or column operation on row opportunity cost matrix. This method is simple one and saves time. (Doing row operation on column opportunity matrix means: Deduct the smallest element in the row from all other elements in the row in column opportunity matrix and vice versa).

The property of total opportunity cost matrix is that it will have at least one zero in every row and column. All the cells, which have zero as the opportunity cost, are eligible for assignment.

Step 5. Once we get the total opportunity **cost matrix, cover all the zeros by MINIMUM NUMBER OF HORIZONTAL AND VERTICAL LINES**. (First cover row or column, which is having maximum number of zeros and then next row or column having next highest number of zeros and so on until all zeros are covered. Remember, only horizontal and vertical lines are to be drawn.

Step 6. If the lines thus drawn are equal to the number of rows or columns (because of square matrix), we can make assignment. If lines drawn are not equal to the number of rows or columns go to step 7.

Step 7. **To make assignment**: Search for a single zero either row wise or column wise. If you start row wise, proceed row by row in search of single zero. Once you find a single zero; assign that cell by enclosing the element of the cell by a **square**. Once all the rows are over, then start column wise and once you find single zero assign that cell and enclose the element of the one cell in a square. Once the assignment is made, then all the zeros in the row and column corresponding to the assigned cell should be cancelled. Continue this procedure until all assignments are made. Some times we may not find single zero and find more than one zero in a row or column. It indicates, that the problem has an alternate solution. We can write alternate solutions. (The situation is known as a **TIE** in assignment problem).

Step 8. If the lines drawn are less than the number of rows or columns, then we cannot make assignment. Hence the following procedure is to be followed:

The cells covered by the lines are known as **Covered cells**. The cells, which are not coveredby lines, are known as **uncovered cells**. The cells at the intersection of horizontal line and vertical lines are known as **Crossed cells.**

(a) Identify the smallest element in the uncovered cells.

(i) Subtract this element from the elements of all other uncovered cells.

(ii) Add this element to the elements of the crossed cells.

(iii) Do not alter the elements of covered cells.

(b) Once again cover all the zeros by **minimum number of horizontal and vertical lines.**

(c) Once the lines drawn are equal to the number of rows or columns, assignment can be made as said in step (6).

(d) If the lines are not equal to number of rows or columns, repeat the steps 7 (a) and 7 (b) untilwe get the number of horizontal and vertical lines drawn are equal to the number of rows orcolumns and make allocations as explained in step (6).

Note: For maximization same procedure is adopted, once we convert the maximization problem into minimization problem by multiplying the matrix by (-1) or by subtracting all the elements of the matrix from highest element in the matrix. Once we do this, the entries in the matrix gives us **relative costs**, hence the problem becomes minimisaton problem. Once we get the optimal assignment, the total value of the original pay off measure can be found by adding the individual original entries for those cells to which assignment have been made.

Now let us take the problem given above and solve.

Solution

Machines (time in hours)

Jobs	*X*	*Y*	*Z*
A	11	16	21
B	20	13	17
C	13	15	12

Step1: To find ROCM.

Machines (time in hours)

Jobs	*X*	*Y*	*Z*
A	0	5	10
B	7	0	4
C	1	3	0

Step 2. To find TOCM (do column operation in ROCM)

Machines (time in hours)

Jobs	X	Y	Z
A	0	5	10
B	7	0	4
C	1	3	0

Because in each column, zero is the lowest element, the matrix remains unchanged, *i.e.* The COCM itself TOCM.

Step 3. To cover all the zeros by minimum number of horizontal and vertical lines.

Machines (time in hours)

Jobs	X	Y	Z
A	0	5	10
B	7	0	4
C	1	3	0

Assignment is:

Machines (time in hours)

Jobs	X	Y	Z
A	**0**	5	10
B	7	**0**	4
C	1	3	**0**

Assignment	*Time in hours.*
A TO X	11
B TO Y	13
C TO Z	12
Total:	36 hours.

Problem 2.

A company has five jobs *V, W, X, Y* and *Z* and five machines *A, B, C, D* and E. The given matrix shows the return in Rs. of assigning a job to a machine. Assign the jobs to machines so as to maximizethe total returns.

Machines. Returns in Rs.

Jobs	*A*	*B*	*C*	*D*	*E*
V	5	11	10	12	4

W	2	4	6	3	5
X	3	12	5	14	6
Y	6	14	4	11	7
Z	7	9	8	12	5

Solution

As the objective function is to maximize the returns, we have to convert the given problem intominimization problem.

Method 1. Here highest element in the matrix is 14, hence subtract all the element form 14 and write the relative costs. (Transformed matrix).

Machines Returns in Rs.

Jobs	*A*	*B*	*C*	*D*	*E*
V	9	3	4	2	10
W	12	10	8	11	9
X	11	2	9	0	8
Y	8	0	10	3	7
Z	7	5	6	2	9

ROCM:

Machines Returns in Rs.

Jobs	*A*	*B*	*C*	*D*	*E*
V	7	1	2	0	8
W	4	2	0	3	1
X	11	2	9	0	8
Y	8	0	10	3	7
Z	5	3	4	0	7

By doing column operation on ROCM, we get the total opportunity cost matrix. TOCM:

Machines Returns in Rs.

Jobs	*A*	*B*	*C*	*D*	*E*
V	3	1	2	0	7
W	0	2	0	3	0
X	7	2	9	0	7
Y	4	0	10	3	6
Z	1	3	4	0	6

Only three lines are there. So we have to go to step 7. The lowest element in uncovered cell is 1, hence subtract 1 from all uncovered cells and add this element to crossed cells and write the matrix. The resultant matrix is:

Machines Return in Rs.

Jobs	*A*	*B*	*C*	*D*	*E*
V	2	0	1	0	6
W	0	3	0	4	0
X	6	1	8	0	6
Y	4	0	10	4	6
Z	0	2	3	0	5

Only foor lines are there, hence repeat the step 7 until we get 5 lines.

Machines Return in Rs.

Jobs	*A*	*B*	*C*	*D*	*E*
V	1	0	0	0	5
W	0	3	0	5	0
X	5	1	7	0	5
Y	3	0	9	4	5
Z	0	3	3	1	5

All zeros are covered by 5 lines, Hence assignment can be made. Start row wise or column wise and go on making assignment, until all assignments are over.

Machines Return in Rs.

Jobs	*A*	*B*	*C*	*D*	*E*
V	2	1	**0**	x0	5
W	1	4	0x	5	**0**
X	6	2	7	**0**	5
Y	3	0	8	3	4
Z	**0**	3	2	0x	4

Job	*Machine*	*Return in Rs.*
V	C	10
W	E	5
X	D	14
Y	B	14
Z	A	7
Total in Rs.		50

Problem 3.

Five jobs are to be assigned to 5 machines to minimize the total time required to process the jobson machines. The times in hours for processing each job on each machine are given in the matrix below. By using assignment algorithm make the assignment for minimizing the time of processing.

Machines (time in hours)

Jobs	*V*	*W*	*X*	*Y*	*Z*
A	2	4	3	5	4
B	7	4	6	8	4
C	2	9	8	10	4
D	8	6	12	7	4
E	2	8	5	8	8

Solution

COCM

Machines (time in hours)

Jobs	*V*	*W*	*X*	*Y*	*Z*
A	2	4	3	5	4
B	7	4	6	8	4
C	2	9	8	10	4
D	8	6	12	7	4
E	2	8	5	8	8

Machines (time in hours)

Jobs	*V*	*W*	*X*	*Y*	*Z*
A	0	0	0	0	0
B	5	0	3	3	0
C	0	5	5	5	0
D	6	2	9	2	0
E	0	4	2	3	4

As the COCM has at least one zero in every column and row, this itself can be considered as TOCM, because as the zero is the lowest number in each column, the matrix remains unchanged. If we cover all the zeros by drawing horizontal and vertical lines, we get only four lines. Applying step 7 weget the following matrix.

Machines (time in hours)

Jobs	*V*	*W*	*X*	*Y*	*Z*
A	2	0	0	0	2
B	7	0	3	3	2
C	0	3	3	3	0
D	6	0	7	0	0
E	0	2	0	1	4

As there are five lines that cover all zeros, we can make assignment.

Jobs	*V*	*W*	*X*	*Y*	*Z*
A	2	~~0~~	**0**	~~0~~	2
B	7	**0**	3	3	2
C	~~0~~	3	3	3	**0**
D	6	~~0~~	7	**0**	~~0~~
E	**0**	2	~~0~~	1	4

Alternate solution:

Machines (time in hours)

Jobs	*V*	~~*W*~~	~~*X*~~	*Y*	*Z*
A	2	0	0	**0**	2
B	7	**0**	3	3	2
C	**0**	3	3	3	~~0~~
D	6	~~0~~	7	~~0~~	**0**
E	~~0~~	2	**0**	1	4

First Solution: *A* to *X*, *B* to *W*, *C* to *Z*, *D* to *Y* and *E* to *V* Cost is: 3 + 4 + 4 + 7 + 2 = 20 hours. Second Solution: *A* to*Y*, *B* to *W*, *C* to *V*, *D* to *Z* and *E* to *X*. Cost is: 5 + 4 + 2 + 4 + 5 = 20 Hours.

When there is a tie, make assignment arbitrarily first to one of the zeros and then proceed, we will get the assignment. When there is a tie, there exists an alternate solution.

Problem 4.

A manager has 4 jobs on hand to be assigned to 3 of his clerical staff. Clerical staff differs in efficiency. The efficiency is a measure of time taken by them to do various jobs. The manager wants to assign the duty to his staff, so that the total time taken by the staff should be minimum. The matrix given below shows the time taken by each person to do a particular job. Help the manager in assigningthe jobs to the personnel.

Jobs.	*Men (time taken to do job in hours).*		
	X	Y	Z
A	10	27	16
B	14	28	7
C	36	21	16
D	19	31	21

Solution

The given matrix is unbalanced. To balance the matrix, open a dummy column with time coefficients as zero.
(DC = Dummy column).

Men (Time taken in hours)

	X	*Y*	*Z*	*DC*
A	10	27	16	0
B	14	28	7	0
C	36	21	16	0
D	19	31	21	0

As every row has a zero, we can consider it as ROCM and by doing column operation, we can write TOCM. Now apply step 7.

Men (Time taken in hours).

Jobs	*X*	*Y*	*Z*	*DC*
A	0	6	9	0
B	4	7	0	0
C	26	0	9	0
D	9	10	14	0

Men (Time taken in hours).

Jobs	*X*	*Y*	*Z*	*DC*
A	**0**	6	9	~~0~~
B	4	7	**0**	~~0~~
C	26	**0**	9	0
D	9	10	14	**0**

The assignment is: *A* to *X*, *B* to *Z*, and *C* to *Y* and *D* is not assigned. Total time required is: 10 + 7 + 21 = 38 Hours.

TRAVELING SALESMAN PROBLEM

Just consider how a postman delivers the post to the addressee. He arranges all the letters in an order and starts from the post office and goes from addressee to addressee and finally back to his postoffice. If he does not arrange the posts in an order he may have to travel a long distance to clear all theposts. Similarly, a traveling sales man has to plan his visits. Let us say, he starts from his head office and go round the branch offices and come back to his head office. While traveling he will not visit the branch already visited and he will not come back until he visits all the branches.

There are different types of traveling salesman's problems. One is **cyclic problem**. In this problem,he starts from his head quarters and after visiting all the branches, he will be back to his head quarters.The second one is **Acyclic problem**. In this case, the traveling salesman leaves his head quarters and after visiting the intermediate branches, finally reaches the last branch and stays there. The first type of the problem is solved by Hungarian method or Assignment technique. The second one is solved by Dynamic programming method.

Point to Note: The traveling salesman's problem, where we sequence the cities or branches he has to visit is a SEQUENCING PROBLEM. But the solution is got by Assignment technique. Hence basically, the traveling salesman problem is a SEQUENCING PROBLEM; the objective is to minimize the total distance traveled.

The mathematical statement of the problem is: Decide variable x_{ij} = 1 or 0 for all values of *I* and *j* so as to:

Minimise $$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} \quad \text{for all } i \text{ and } j = 1,2\ldots.n \text{ Subject to}$$

$$\sum_{j=1}^{n} X_{ij} = 1 \text{ for } i = 1,2\ldots.n \quad \text{(Depart from a city once only)}$$

$$\sum_{i=1}^{n} X_{ij} = 1 \text{ for } j = 1,2\ldots.n \quad \text{(Arrive at a city once only)}$$

And all $x_{ij} \geq 0$ for all i and j

This is indeed a statement of assignment problem, which may give to or more disconnected cycles in optimum solution. This is not permitted. That is salesman is not permitted to return to the origin of his tour before visiting all other cities in his itinerary. The mathematical formulation above does not take care of this point.

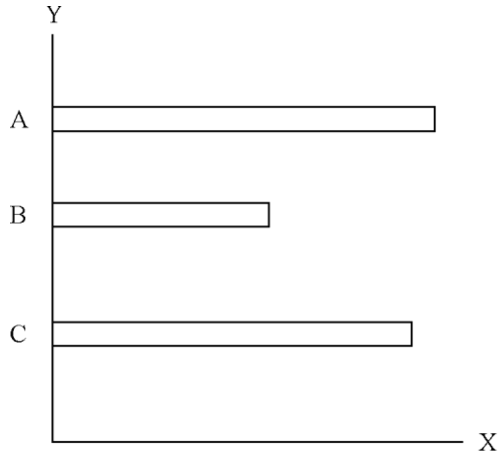
A restriction like $X_{ab} + X_{bc} + X_{ca}$ □ 2 will prevent sub-cycles of cities *A*, *B*, *C* and back to *A*. It is sufficient to state at this stage that all sub- cycles can be ruled out by particular specifications of linear constraints. This part, it is easy to see that a variable x_{ij} = 1, has no meaning. To exclude this from solution, we attribute very large cost to it *i.e.* infinity or big *M*, which is very larger than all the elements in the matrix. In our solutions big M is used.

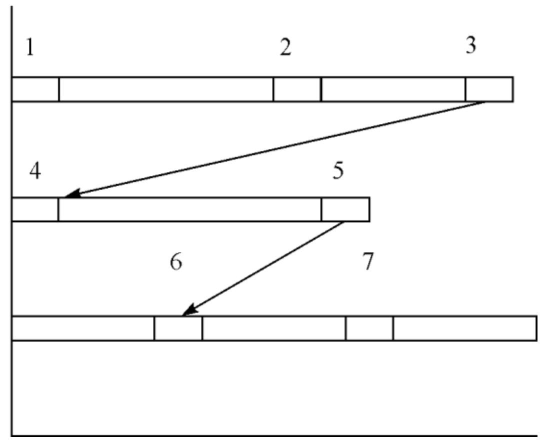
UNIT IV- PROJECT MANAGEMENT AND QUEUING MODELS

INTRODUCTION

Programme Evaluation and Review Technique (PERT) and Critical Path Method (CPM) are two techniques that are widely used in planning and scheduling the large projects. A project is a combination of various activities. For example, Construction of a house can be considered as a project. Similarly, conducting a public meeting may also be considered as a project. In the above examples, construction of a house includes various activities such as searching for a suitable site, arranging the finance, purchase of materials, digging the foundation, construction of superstructure etc. Conducting a meeting includes, printing of invitation cards, distribution of cards, arrangement of platform, chairs for audience etc. In planning and scheduling the activities of large sized projects, the two network techniques — PERT and CPM — are used conveniently to estimate and evaluate the project completion time and control the resources to see that the project is completed within the stipulated time and at minimum possible cost. Many managers, who use the PERT and CPM techniques, have claimed that these techniques drastically reduce the project completion time. But it is wrong to think that network analysis is a solution to all bad management problems. In the present chapter, let us discuss how PERT and CPM are used to schedulethe projects.

Initially, projects were represented by **milestone chart** and **bar chart**. But they had little use in controlling the project activities. **Bar chart** simply represents each activity by bars of length equal to the time taken on a common time scale as shown in figure 15. l. This chart does not show interrelationship between activities. It is very difficult to show the progress of work in these charts. An improvement in bar charts is **milestone chart**. In milestone chart, key events of activities are identified and each activity is connected to its preceding and succeeding activities to show the logical relationship between activities. Here each key event is represented by a node (a circle) and arrows instead of bars represent activities, as shown in figure. The extension of milestone chart is PERT and CPM network methods.





PERT AND CPM

In PERT and CPM the milestones are represented as ***events***. Event or node is either starting of an activity or ending of an activity. Activity is represented by means of an arrow, which is resource consuming. Activity consumes resources like time, money and materials. Event will not consume any resource, but it simply represents either starting or ending of an activity. Event can also be represented by rectangles or triangles. When all activities and events in a project are connected logically and sequentially, they form a ***network***, which is the basic document in network-based management. The basic steps for writing a network are:

(a) List out all the activities involved in a project. Say, for example, in building construction, the activities are:
   (i) Site selection,
   (ii) Arrangement of Finance,
   (iii) Preparation of building plan,
   (iv) Approval of plan by municipal authorities,
   (v) Purchase of materials,
   (vi) Digging of foundation,
   (vii) Filling up of foundation,
   (viii) Building superstructure,
   (ix) Fixing up of doorframes and window frames,
   (x) Roofing,
   (xi) Plastering,
   (xii) Flooring,
   (xiii) Electricity and water fittings,
   (xiv) Finishing.

(b) Once the activities are listed, they are arranged in sequential manner and in logical order. For example, foundation digging should come before foundation filling and so on.

(c) After arranging the activities in a logical sequence, their time is estimated and written against each activity. For example: Foundation digging: 10 days, or 1½ weeks.

(d) Some of the activities do not have any logical relationship, in such cases; we can start those activities simultaneously. For example, foundation digging and purchase of materials do not have any logical relationship. Hence both of them can be started simultaneously. Suppose foundation digging takes 10 days and purchase of materials takes 7 days, both of them can be finished in 10 days. And the successive activity, say foundation filling, which has logical relationship with both of the above, can be started after 10 days. Otherwise, foundation digging and purchase of materials are done one after the other; filling of foundation should be started after 17 days.

(e) Activities are added to the network, depending upon the logical relationship to complete the project network.

Some of the points to be remembered while drawing the network are

(a) There must be only one beginning and one end for the network, as shown in figure 3.

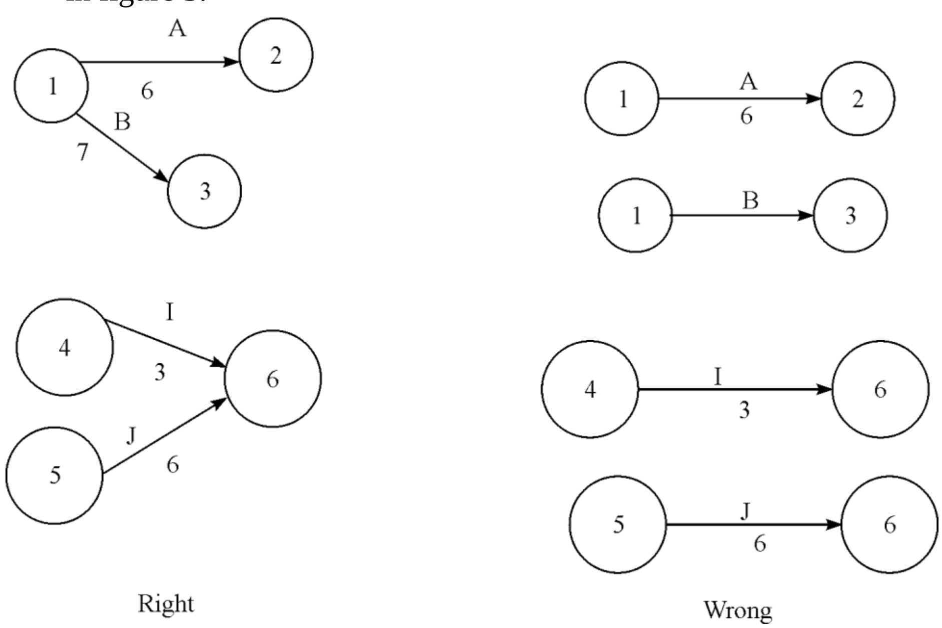
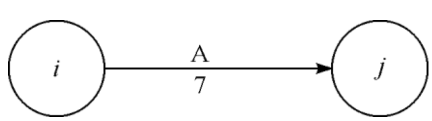


Figure . 3. Writing the network.

(b) Event number should be written inside the circle or node (or triangle/square/rectangle etc). Activity name should be capital alphabetical letters and would be written above the arrow. The time required for the



activity should be written below the arrow as in figure 4

Figure 4. Numbering and naming the activities.

(a) While writing network, see that activities should not cross each other. And arcs or loops as in figure 5 should not join Activities.

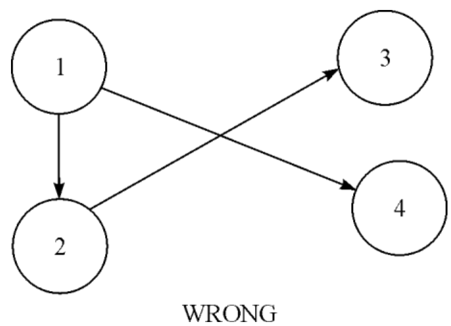


Figure 5. Crossing of activities not allowed.

(b) While writing network, looping should be avoided. This is to say that the network arrows should move in one direction, *i.e.* starting from the beginning should move towards the end,as in figure 6.

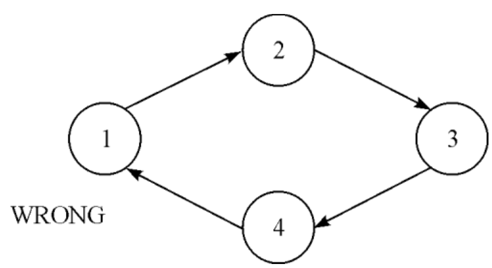


Figure 6. Looping is not allowed.

(c) When two activities start at the same event and end at the same event, they should be shownby means of a **dummy activity** as in figure 7. Dummy activity is an activity, which simply shows the logical relationship and does not consume any resource. It should be represented by a dotted line as shown. In the figure, activities *C* and *D* start at the event 3 and end at event 4. *C* and *D* are shown in full lines, whereas the dummy activity is shown indotted line.

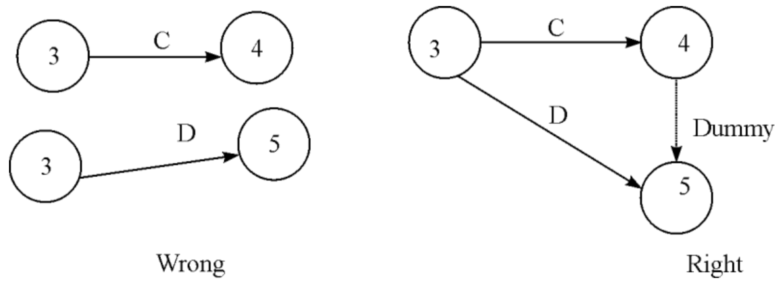


Figure 7. Use of Dummy activity.

(a) When the event is written at the tail end of an arrow, it is known as **tail event**. If event is written on the head side of the arrow it is known as **head event**. A tail event may have any number of arrows (activities) emerging from it. This is to say that an event may be a tail event to any number of activities. Similarly, a head event may be a head event for any number of activities. This is to say that many activities may conclude at one event. This is shown in figure 8.

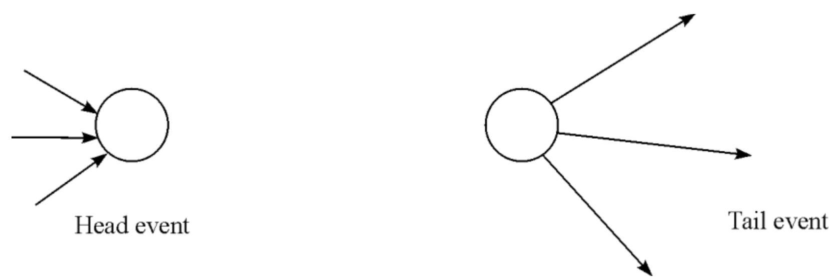


Figure 8. Tail event and Head event.

The academic differences between PERT network and CPM network are:

(*i*) PERT is event oriented and CPM is activity oriented. This is to say that while discussing about PERT network, we say that Activity 1-2, Activity 2-3 and so on. Or event 2 occurs after event 1 and event 5 occurs after event 3 and so on. While discussing CPM network, we say that Activity *A* follows activity *B* and activity *C* follows activity *B* and so on. Referring to the network shown in figure 9, we can discuss as under.

PERT way: Event 1 is the predecessor to event 2 or event 2 is the successor to event

1. Events 3 and 4 are successors to event 2 or event 2 is the predecessor to events 3 and 4.

CPM way: Activity 1-2 is the predecessor to Activities 2-3 and 2-4 or Activities 2-3 and 2-4 are the successors to activity 1-2.

ii) PERT activities are probabilistic in nature. The time required to complete the PERT activity cannot be specified correctly. Because of uncertainties in carrying out the activity, the time cannot be specified correctly. Say, for example, if you ask a contractor how much time it takes to construct the house, he may answer you that it may take 5 to 6 months. This is because of his expectation of uncertainty in carrying out each one of the activities in the construction of the house. Another example is if somebody asks you how much time you require to reach railway station from your house, you may say that it may take 1 to 1½ hours. This is because you may think that you may not get a transport facility in time. Or on the way to station, you may come across certain work, which may cause delay in your journey from house to station. Hence PERT network is used when the activity times are probabilistic.

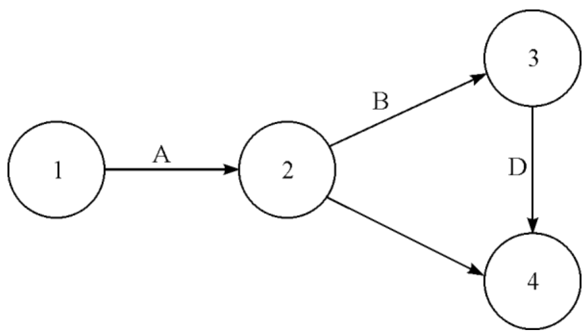


Figure .9. Logical relationship in PERT and CPM.

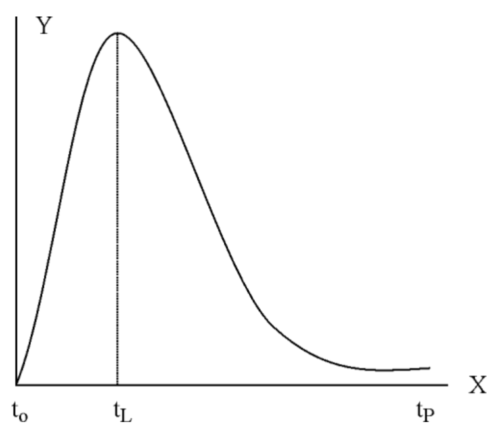


Figure 10. Three Time estimates.

There are three time estimates in PERT, they are:

(a) **OPTIMISTIC TIME:** Optimistic time is represented by $\mathbf{t_O}$. Here the estimator thinks that everything goes on well and he will not come across any sort of uncertainties and estimates lowest time as far as possible. He is optimistic in his thinking.

(b) **PESSIMISTIC TIME:** This is represented by $\mathbf{t}_P$. Here estimator thinks that everything goes wrong and expects all sorts of uncertainties and estimates highest possible time. He is pessimistic in his thinking.

(c) **LIKELY TIME:** This is represented by t_L. This time is in between optimistic and pessimistic times. Here the estimator expects he may come across some sort of uncertainties and many a time the things will go right.

So while estimating the time for a PERT activity, the estimator will give the three time estimates. When these three estimates are plotted on a graph, the probability distribution that we get is closely associated with **Beta Distribution curve**. For a Beta distribution curve as shown in figure 6.10, the characteristics are:

Standard deviation = $(t_P - t_O)/_6$ = □ , $t_P - t_O$ is known as range.

Variance = $\{(t_P - t_O)/6\}^2$ = □ ^{2}

Expected Time or Average Time = $t_E = (t_O + 4t_L + t_P) / 6$

These equations are very important in the calculation of PERT times. Hence the student hasto remember these formulae.

Now let us see how to deal with the PERT problems.

(*g*) **Numbering of events:** Once the network is drawn the events are to be numbered. In PERT network, as the activities are given in terms of events, we may not experience difficulty. Best in case of CPM network, as the activities are specified by their name, is we have to number the events. For numbering of events, we use D.R. Fulkerson's rule. As per this rule:

An initial event is an event, which has only outgoing arrows from it and no arrow enters it. Number that event as 1.

Delete all arrows coming from event 1. This will create at least one more initial event.Number these initial events as 2, 3 etc.

Delete all the outgoing arrows from the numbered element and which will create some more initial events. Number these events as discussed above.

Continue this until you reach the last event, which has only incoming arrows and no outgoing arrows.

While numbering, one should not use negative numbers and the initial event should not be assigned 'zero'. When the project is considerably large, at the time of execution of the project, theproject manager may come to know that some of the activities have been forgotten and they are tobe shown in the current network. In such cases, if we use **skip numbering**, it will be helpful. Skip numbering means, skipping of some numbers and these numbers may be made use to represent the events forgotten. We can skip off numbers like 5, 10, 15 etc. or 10, 20 and 30 or 2, 12, 22 etc. Another way of numbering the network is to start with 10 and the second event is 20 and so on. This is a better way of numbering the events.

Let now see how to write network and find the project completion time by solving some typical problems.

Problem 1.

A project consists of 9 activities and the three time estimates are given below. Find the project completion time (T_E).

1. Write the network for the given project and find the project completion time?

Activities		*Days*		
i	*j*	T_O	T_L	T_P
10	20	5	12	17
10	30	8	10	13
10	40	9	11	12
20	30	5	8	9
20	50	9	11	13
40	60	14	18	22
30	70	21	25	30
60	70	8	13	17
60	80	14	17	21
70	80	6	9	12

Solution

In PERT network, it is easy to write network diagram, because the successor and predecessor event relationships can easily be identified. While calculating the project completion time, we have to calculate t_e *i.e.* expected completion time for each activity from the given three-time estimates. In case, we calculate project completion time by using t_O or t_L or t_P separately, we will have three completion times. Hence it is advisable to calculate t_E expected completion time for each activity and then the project completion time. Now let us work out expected project completion time.

Predecessor event	*Successor event*	*Time in days*			$T_E = (t_O + 4t_L + t_P)/6$	*Range* $t_P - t_O$	*S.D* (σ) $(t_P - t_O)/6$	*Variance* σ^2
10	20	5	12	17	9.66 (10)	12	2	4

10	30	8	10	13	10.17 (10)	5	0.83	0.69
10	40	9	11	12	10.83 (11)	3	0.5	0.25
20	30	5	8	9	7.67 (8)	4	0.66	0.44
20	50	9	11	13	11.00 (11)	4	0.66	0.44
40	60	14	18	22	18.00 (18)	8	1.33	1.78
30	70	21	25	30	25.18 (25)	9	1.5	2.25
60	70	8	13	17	12.83 (13)	9	1.5	2.25
50	80	14	17	21	17.17 (17)	7	1.16	1.36
70	80	6	9	12	9.00 (9)	6	1.0	1.0

For the purpose of convenience the t_E got by calculation may be rounded off to nearest whole number (the same should be clearly mentioned in the table). The round off time is shown in sbrackets. In this book, in the problems, the decimal, will be rounded off to nearest whole number.

To write the network program, start from the beginning *i.e.* we have 10 – 20, 10 – 30 and 10 –

40. Therefore from the node 10, three arrows emerge. They are 10 – 20, 10 – 30 and 10 – 40. Next from the node 20, two arrows emerge and they are 20 – 30 and 20 – 50. Likewise the network is constructed. The following convention is used in writing network in this book.

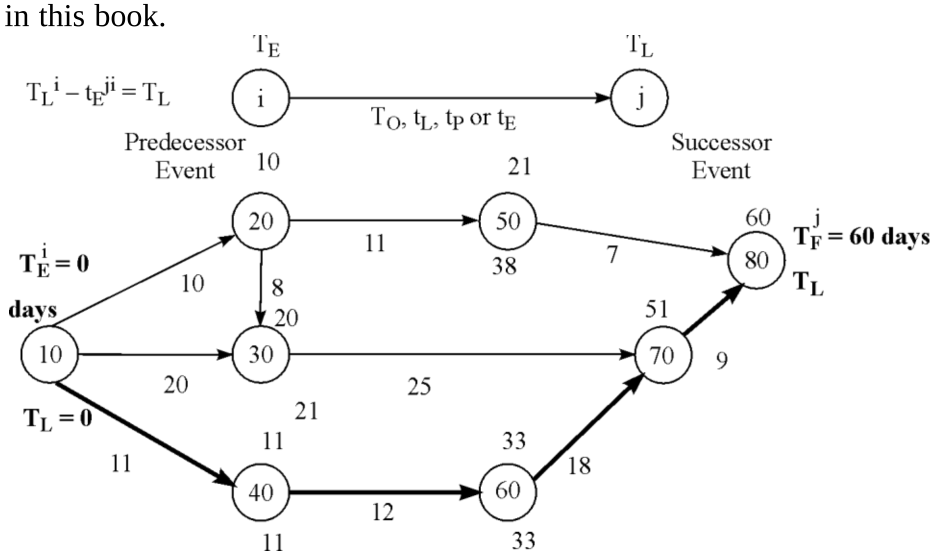


Figure 11. Network for Problem 1

Let us start the event 10 at 0th time *i.e.* expected time $T_E = 0$. Here T_E represents the occurrence time of the event, whereas t_E is the duration taken by the activities. T_E belongs to event, and t_E belongs to activity.

$$T_E^{10} = 0$$

$T_E^{20} = T_E^{10} + t_E^{10-20} = 0 + 10 =$

10 days $T_E^{30} = T_E^{10} + t_E^{10-30} =$

0 + 10 = 10 days $T_E^{30} = T_E^{20}$ +
t_E^{20-30} = 10 + 8 = 18 days

The event 30 will occur only after completion of activities 20–30 and 10–30. There are two routes to event 30. In the **forward pass** *i.e.* when we start calculation from 1st event and proceed through last event, we have to workout the times for all routes and select the **highest one** and the **reverse** is the case of the **backward pass** *i.e.* we start from the last event and work back to the first event to find out the occurrence time.

$T_E^{40} = T_E^{10} + t_E^{10-40}$ = 0 + 11 =
11 days $T_E^{50} = T_E^{20} + t_E^{20-30}$ =
10 + 11 = 21 days $T_E^{60} = T_E^{40}$+
t_E^{40-60} = 11 + 18 = 29 days T_E^{70}
$= T_E^{30} + t_E^{30-70}$ = 18 + 25 = 43
days $T_E^{70} = T_E^{60} + t_E^{60-70}$ = 29
+ 13 = 42 days $T_E^{80} = T_E^{70} + t_E^{70}$
$- 80$ = 43 + 9 = 52 days T_E^{80} =
$T_E^{50} + t_E^{50-80}$ = 21 + 17 = 38
days

T_E^{80} = 52 days. Hence the project completion time is 52 days. The path that gives us 52 days is known as

Critical path. Hence 10–20–30–70–80 is the critical path. Critical path may be represented by double line ($\Rightarrow$) or thick line ($\longrightarrow$) or hatched line ($\nrightarrow$). In this book thick line is used. All other parts *i.e.* 10–40–60–70–80, 10–20–50–80 and 10–30–70–80 are known as **non-critical paths**. All activities on critical path are **critical activities**.

The significance of critical path is delay in completion of critical activities which will increase the project completion time.

Now in the above project, the project completion time is 52 days. In case everything goes correctly the project will be completed in 52 days. Suppose the manager may want to completed the project in 50 days, then what is the probability of completing the project in 50 days? To find the answer for this, let us recollect what is discussed earlier.

Activity $i - j$ is given three time estimates *i.e.* t_O, t_L, and t_P and assumed that the distribution of these time estimates follows ß distribution curve. The approximate mean time for each activity is given by

$$t_E = (t_O + 4t_L + t_P)/6$$

The meaning of this expected time is that there is a fifty-fifty chance of completing the activities in a time duration t_E as shown in the curve.

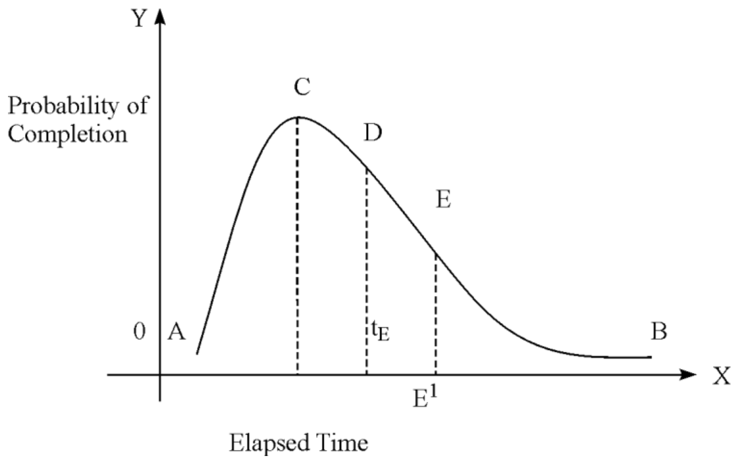


Figure 12

The vertical line at *D* represents t_E and the chance of completing activities at t_E is ½ Suppose wewant to find out probabilities of completing the activities at *EE* ?,

The probability = Area under ACE / Area under ACB

While calculating the probability of completing the project (having number of activities), the following procedure is applied (here, we apply central limit theorem).

Step 1: Identify critical path and critical activitiesStep 2: Find variance (□^{2}) for critical activities.

$$\square_{ij}^{2} = [\,(t_{p}^{ij} - t_{o}^{ij})/6]^{2}$$

Step 3: List out critical activities and their □^{2}

Step 4: Find the sum of variance of critical activities *i.e.* □□^{2}

Step 5: Find the square root of sum of variance √ *i.e.*

Step 6: Find the difference between the contractual time (T_L) *i.e.*, time by which the project is to becompleted and project completion time T_E, i.e. $T_L - T_E$

Depending on the value of TL, this may be +ve or 0 or –ve number. That is If TL = TE then TL – TE = 0

If TL > TE then TL – TE = Positive Number

If TL < TE then TL – TE = Negative Number

Step 7: Find the ratio (TL – TE)/ □□2 = Z, this is the length of ordinate at TL on the curve.

Step 8: Refer to Table 15.1, which gives the height of Z and the probability of completing the project.

If TL = TE the probability is ½.

If TL > TE then Z is Positive, the probability of completing the project is higher than 0.5 If TL < TE then Z is Negative, the probability of completing the project is lower than 0.5

Table: I

Standard Normal Distribution Function

Z (+)	*Probability P_r (%)*	*Z (–)*	*Probability (P_r) (%)*
0	50.0	0	50.0
+0.1	53.98	–0.1	46.02
+0.2	57.95	–0.2	42.07
+0.3	61.79	–0.3	38.21
+0.4	65.54	–0.4	34.46
+0.5	69.15	–0.5	30.85
+0.6	72.57	–0.6	27.43
+0.7	75.80	–0.7	24.20
+0.8	78.81	–0.8	21.19
+0.9	81.59	–0.9	18.41
+1.0	84.13	–1.0	15.87
+1.1	86.43	–1.1	13.57
+1.2	88.49	–1.2	11.51
+1.3	90.32	–1.3	9.68
+1.4	91.92	–1.4	8.08
+1.5	93.32	–1.5	6.68
+1.6	94.52	–1.6	5.48
+1.7	95.54	–1.7	4.46
+1.8	96.41	–1.8	3.59
+1.9	97.13	–1.9	2.87
+2.1	98.21	–2.1	1.79
+2.2	96.61	–2.2	1.39
+2.3	98.93	–2.3	1.07
+2.4	99.19	–2.4	0.82
+2.5	99.38	–2.5	0.62
+2.6	99.53	–2.6	0.47
+2.7	99.65	–2.7	0.35
+2.8	99.74	–2.8	0.26
+2.9	99.81	–2.9	0.19
+3.0	99.87	–3.0	0.13

Now coming to the problem Number 15.1, given that T_L = 52 days.

Critical activities		σ^2
I	J	
10	20	4.00
20	30	0.44
30	70	2.25
70	80	1.00
$\Sigma\sigma^2$		7.69

$$\sqrt{\Sigma\sigma^2} = \sqrt{7.69} = 2.77$$

$$T_L - T_E = 50 - 52 = -2$$

$$(T_L - T_E) / \sqrt{\Sigma\sigma^2} = -2 / 2.77 = -0.722 = Z = \text{Normal deviate.}$$

–0.722 falls at between probability 22.7. The probability is very low. Hence the manager shouldnot accept to complete the project in 50 days.

Say for example given that T_L = 58 days then 58 – 55 = +3

$$(T_L - T_E) / \sqrt{\Sigma\sigma^2} = 3 / 2.77 = 1.08 = Z = \text{Normal deviate.}$$

1.08 falls at 85% probability. The probability of completing the project is high. The manager canaccept the offer.

Let us continue further discussion on problem no. 1

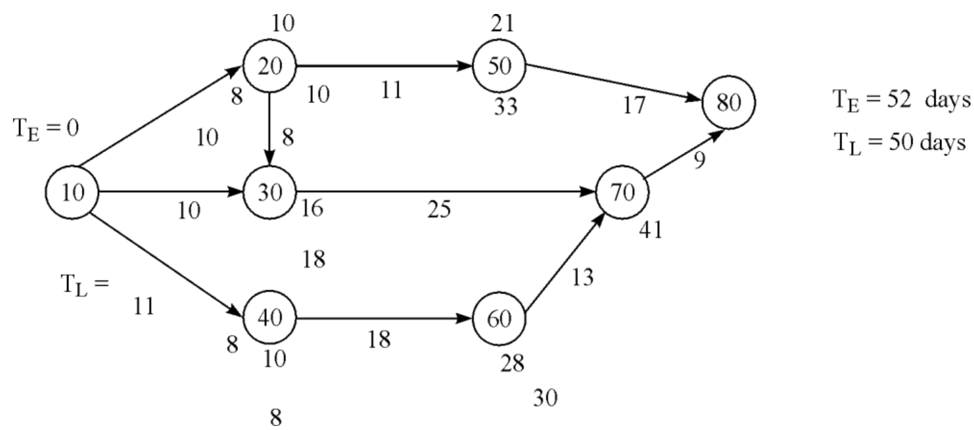


Figure 13.

Let us assume that the contractual time = 50 days.

This is written at the end event. Now let us work back to find out when the project should be started if the delivery time is 52 days.

$$T_L^{80} = 50 \text{ days}$$

$T_L^{70} = T_L^{80} - t_E^{70-80} = 50 - 9 = 41$ days this we write below the node

$$T_L^{50} = T_L^{80} - t_E^{50-80} = 50 - 17 = 33 \text{ days}$$

$$T_L^{60} = T_L^{70} - t_E^{60-70} = 41 - 13 = 28 \text{ days}$$

$$T_L^{40} = T_L^{60} - t_E^{40-60} = 28 - 22 = 6 \text{ days}$$

$$T_L^{20} = T_L^{50} - t_E^{20-50} = 33 - 11 = 22 \text{ days}$$

$$T_L^{30} = T_L^{70} - t_E^{30-70} = 41 - 25 = 16 \text{ days}$$

$$T_L^{20} = T_L^{30} - t_E^{20-30} = 16 - 8 = 8 \text{ days}$$

T_L^{20} has two values *i.e.* 22 days and 8 days. Here as we are going back to find out when the project is to be started, take lowest of the two *i.e.* $T_L^{20} = 8$ days

$T_L^{10} = T_L^{20} - t_E^{10-20} = 8 - 12 = -4$ days

$$T_L^{10} = T_L^{30} - t_E^{10-30} = 16 - 10 = 6 \text{ days}$$

$$T_L^{10} = T^{40} - t^{10-40} = 10 - 11 = -1 \text{ days}$$

Take T_L = –1 days which is lowest. Hence the project is to be started 1 day before the scheduled starting time.

Now at the critical events calculate ($T_L - T_E$). For all critical events it is –1 day.

This ($T_L - T_E$) is known as slack and is represented by Greek letter 'ι'. On the critical path ι remains to be same. In fact slack is the breathing time for the contractor. If ($T_L - T_E$), slack for all critical events is zero. If ($T_L > T_E$) it is a positive number and if ($T_L < T_E$) it will be a negative number for all critical events. For non-critical activities this difference between T_L and T_E *i.e.* ($T_L - T_E$) shows the breathing time available to the manager at that activity. For example take the event 50

For this event T_L = 33days and T_E = 21 days *i.e.* 33 – 21 = 12 days of time available for this manager. In case of any inconvenience he can start the activity 50-80 any day between 21st day and 33rd day. Now let us work out some more examples.

Problem 2.

Steps involved in executing an order for a large engine generator set are given below in a jumbled manner. Arrange them in a logical sequence, draw a PERT network and find the expected execution time period.

Activities (not in logical order)	**Time in weeks**		
	t_O	t_L	t_P
Order and receive engine	1	2	3
Prepare assembly drawings	1	1	1
Receive and study order	1	2	3
Apply and receive import license for generator	3	5	7
Order and receive generator	2	3	5
Study enquiry for engine generator set	1	2	3
Fabricate switch board	2	3	5
Import engine	1	1	1
Assemble engine generator	1	2	3
Submit quotation with drawing and full	1	2	3
Prepare base and completing	2	3	4
Import generator	1	1	1
Order and receive meters, switch gears for switch board	2	3	4
Test assembly	1	1	1

Solution

As the activities given in the problem are not in logical order, first we have to arrange them in a logical manner.

S.No.	*Activities*	*Time in weeks*		
		t_O	t_L	t_P
A	Study enquiry for engine generator set	1	2	3
B	Prepare assembly drawings	1	1	1
C	Submit quotation with drawing and full	1	2	3
D	Receive and study order	1	2	3
E	Apply and receive import license for generator	3	5	7
F	Order and receive engine	1	2	3
G	Order and receive generator	2	3	5
H	Inspect engine	1	1	1
I	Order and receive meters, switch gears for switch board	2	3	4
J	Prepare base	2	3	4
K	Complete assemble engine generator	1	2	3
L	Fabricate switch board	2	3	5
M	Test assembly	1	1	1

Solution:

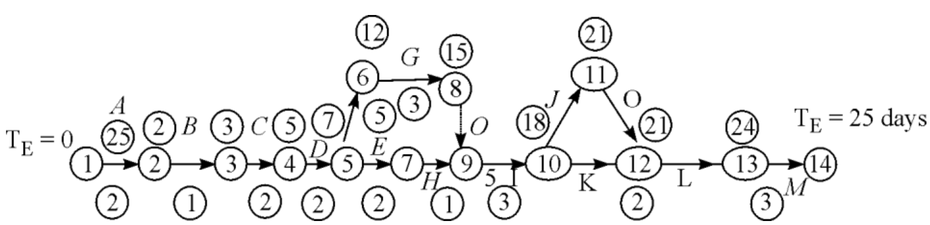


Figure 14.

The second step is to write network and number the events

Activities	*Predecessor event*	*Successor Event*	*Weeks* t_O	t_L	t_P	$t_E = \frac{t_O + 4t_L + t_P}{6}$	□ = $(t_P - t_O)/6$	□^{2}
A	1	2	1	2	3	2	1/3 = 0.33	0.102
B	2	3	1	1	1	1	0	0
C	3	4	1	2	3	2	0.33	0.102
D	4	5	1	2	3	2	0.33	0.102
E	4	6	3	5	7	5	0.66	0.44
F	4	7	1	2	3	2	0.33	0.102
G	6	9	2	3	5	3	0.5	0.25
H	7	10	1	1	1	1	0	0
I	5	8	2	3	4	3	0.33	0.102
J	10	11	2	3	4	3	0.33	0.102
K	11	12	1	2	3	2	0.33	0.102
L	8	12	2	3	5	3	0.5	0.25
M	12	13	1	1	1	1	0	0

CRITICAL PATH = 1 – 2 –3 – 4 – 6 – 8 – 9 – 10 – 11 – 12 – 13
T_E = 25 Weeks

Problem 3.

A small project is composed of 7 activities whose time estimates are listed below. Activities are being identified by their beginning (*i*) and ending (*j*) node numbers.

Activities		*Time in weeks*		
i	*j*	t_o	t_l	t_p
1	2	1	1	7
1	3	1	4	7
1	4	2	2	8
2	5	1	1	1
3	5	2	5	14
4	6	2	5	8
5	6	3	6	15

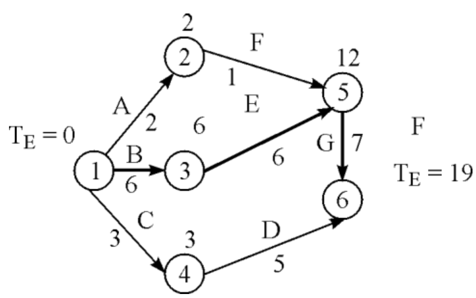
1. Draw the network
2. Calculate the expected variances for each
3. Find the expected project completed time
4. Calculate the probability that the project will be completed at least 3 weeks than

expected

5. If the project due date is 18 weeks, what is the probability of not meeting the due date?

Solution

Activities		*Weeks*			t_E = t_E + 4t_L+ t_P /6	t_E	□ =(t_p – t_o)/6	□2
i	*j*	t_O	t_L	t_P				
1	2	1	1	7	2	6	1	1
1	3	1	4	7	6	6	1	**1**
1	4	2	2	8	3	6	1	1
2	5	1	1	1	1	0	0	0
3	5	2	5	14	6	12	2	**4**
4	6	2	5	8	5	6	1	1
5	6	3	6	15	7	12	2	**4**



Critical activities	*Variance*
1 – 3	1
3 - 5	4
5 - 6	4
$\Sigma\sigma^2$	9

$\sqrt{\Sigma\sigma^2} = \sqrt{9}$ □ 3

4. Probability of completing the project at least 3 weeks earlier i.e. 16 in weeks

T_L = 16 weeks, T_E = 19 weeks.

$T_L - T_E = -3$ weeks

$Z = (T_L - T_E) / \sqrt{\Sigma\sigma^2}$

$= -3 / 3 = -1$

From table the

eting the project = 15.9%

4. if T_L = 18 weeks. Probability of completing in 11 weeks is (18 – 19) / 3 = –1/3From table the probability = 38.2%

Probability of not meeting due date = 100 – 38.2 = 61.8%

i.e. 61.8% of the time the manager cannot complete the project by due date.

Example 4

There are seven activities in a project and the time estimates are as follows

Activities	*Time in weeks*		
	t_O	t_L	t_P
A	2	6	10
B	4	6	12
C	2	3	4
D	2	4	6
E	3	6	9
F	6	10	14
G	1	3	5

The logical of activities are:

1. Activities *A* and *B* start at the beginning of the project.
2. When *A* is completed *C* and *D* start.
3. *E* can start when *B* and *D* are finished.
4. *F* can start when *B*, *C* and *D* are completed and is the final activity.
5. *G* can start when *F* is finished and is final activity the.

(a) What is the expected time of the duration of the project?

(b) What is the probability that project will be completed in 22 weeks?

Solution

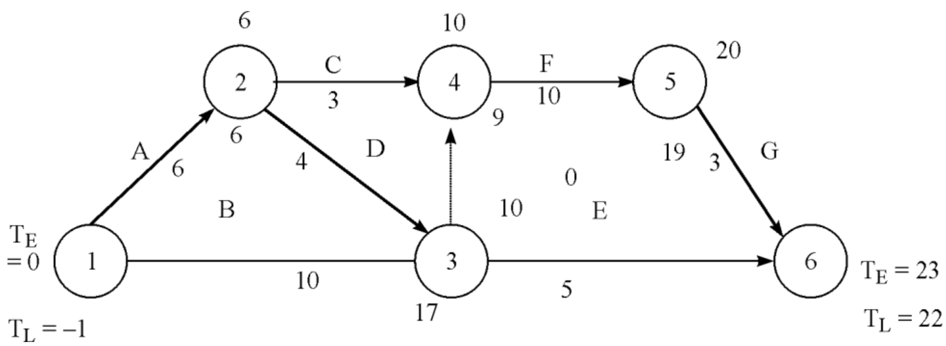
First we use to establish predecessor and successor relationship and then find standard deviation

□ , variance □ ^{2} and expected time of completing activities, t_E.

Activities	*Predecess or Event*	*Weeks*			t_E = t_O + 4t_L+ t_P /6	□ = (t_P – t_O)/6	□*2*
		t_O	t_L	t_P			
A	–	2	6	10	6	8/6 = 1.33	1.77
B	–	4	6	12	10	8/6 = 1.33	1.77
C	A	2	3	4	3	2/6 = 0.33	0.11

D	A	2	4	6	4	4/6 = 0.66	0.44
E	B, D	3	6	9	5		1.
F	B, C, D	6	10	14	10	8/6 = 1.33	1.77
G	F	1	3	5	3	4/6=0.66	0.44

Now to write network the logical (predecessor) relationship is considered.



After writing the network, numbering of events and t_E is entered on the network. Next the project completion time is worked out. The project completion time T_E = 23

Critical path	*Variance* σ^2
B	1.77
F	1.77
G	0.44
$\Sigma\sigma^2$	3.98

weeks. This project has two critical paths i.e. $A - D - F - G$ and $B - F - G$.

Critical path	*Variance* σ^2
A	1.77
D	0.44
F	1.77
G	0.44
$\Sigma\sigma^2$	4.42

In the problem T_L is given as 22 weeks. Therefore $T_L - T_E = 22 - 23$
$= -1$Therefore probability of completing the project in 22 weeks

$-1 / 2.10 = -0.476$ OR $-1 / 1.99 = 0.502$

The probability of completing the project is approximately 49%.

CRITICAL PATH METHOD (CPM) FOR CALCULATINGPROJECT COMPLETION TIME

In critical path method, the time duration of activity is deterministic in nature *i.e*. there will be a single time, rather than three time estimates as in PERT networks. The network is activity oriented. The three ways in which the CPM type of networks differ from PERT networks are

CPM	PERT
(*a*) Network is constructed on the basis of jobs or activities (activity oriented).	(*a*) Network is constructed basing on the events (event oriented)
(*b*) CPM does not take uncertainties involved in the estimation of times. The time required is deterministic and hence only one time is considered.	(*b*) PERT network deals with uncertainties and hence three time estimations are considered (Optimistic Time, Most Likely Time and Pessimistic Time)
(*c*) CPM times are related to cost. That is can be by decreasing the activity duration direct costs increased (crashing of activity duration is possible)	(*c*) As there is no certainty of time, activity duration cannot be reduced. Hence cost cannot be expressed correctly. We can say expected cost of completion of activity (crashing of activity duration is not possible)

Writing the CPM Network

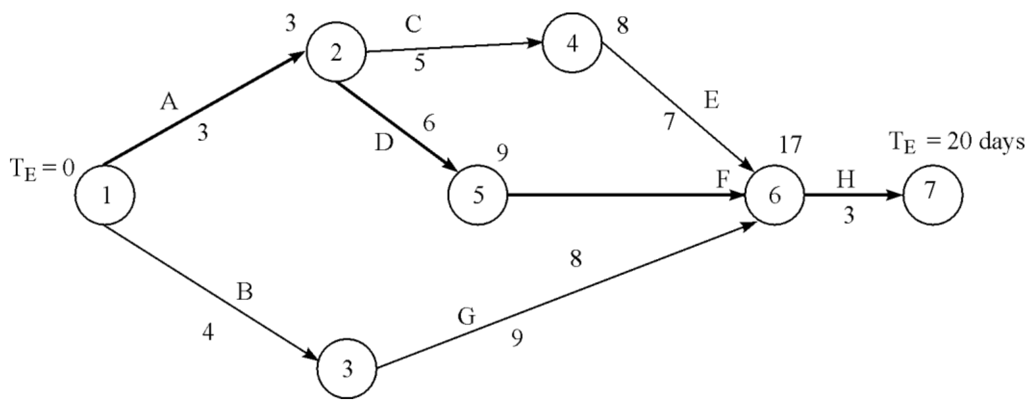
First, one has to establish the logical relationship between activities. That is predecessor and successor relationship, which activity is to be started after a certain activity. By means of problems let us see how to deal with CPM network and the calculations needed.

Problem 5.

A company manufacturing plant and equipment for chemical processing is in the process of quoting tender called by public sector undertaking. Help the manager to find the project completion time to participate in the tender.

S.No.	*Activities*		*Days*
1	A	–	3
2	B	–	4

3	C	A	5
4	D	A	6
5	E	C	7
6	F	D	8
7	G	B	9
8	H	E, F, G	3



(1) Write the network referring to the data

(2) Number the events as discussed earlier.

(3) Calculate T_E as done in PERT network $T_E^j = (T_E^i + T^{ij})]$

(4) Identify the critical path

Project completion time = 20 weeks and the critical path = *A – D – F – H.*

Problem 6.

A small project has 7 activities and the time in days for each activity is given below:

Activity	*Duration in days*
A	6
B	8
C	3
D	4
E	6
F	10
G	3

Given that activities *A* and *B* can start at the beginning of the project. When *A* is completed *C* and *D* can start. *E* can start only when *B* and *D* are finished. *F* can start when *B, C* and *D* are completed and is the final activity. *G* can start when *E* is finished and is the final activity. Draw the network and find the project completion time.

Activity	Immediate predecessor	Time in days
A	–	6
B	–	8
C	A	3
D	A	4
E	B , D	6
F	B, C or D	10
G	E	3

Draw the network and enter the times and find T_E .

Solution

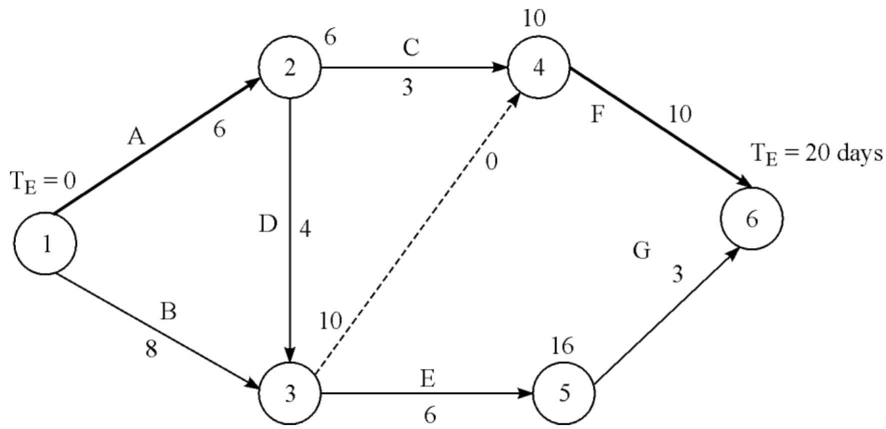


Figure 18

Project completion time = 20 days and critical path is *A – D – F*.

QUEUING MODELS

INTRODUCTION

Before going to ***waiting line theory or queuing theory***, one has to understand two things in clear. They are ***service and customer or element.*** Here customer or element represents a person or machine or any other thing, which is in need of some service from servicing point. Service represents any type of attention to the customer to satisfy his need. For example,

1. Person going to hospital to get medical advice from the doctor is an element or a customer,
2. A person going to railway station or a bus station to purchase a ticket for the journey is a customer or an element,
3. A person at ticket counter of a cinema hall is an element or a customer,
4. A person at a grocery shop to purchase consumables is an element or a customer,

5. A bank pass book tendered to a bank clerk for withdrawal of money is an element or a customer,
6. A machine break down and waiting for the attention of a maintenance crew is an element ora customer.
7. Vehicles waiting at traffic signal are elements or customers,
8. A train waiting at outer signal for green signal is an element or a

customerLike this we can give thousands of examples.

In the above cases, the service means,

1. Doctor is a service facility and medical care is a service,
2. Ticket counter is a service facility and issue of ticket is service.
3. Ticket counter is a service facility and issue of ticket is service.
4. Shop owner is a service facility and issue of items is service.
5. Bank clerk is a service facility and passing the cheque is service.
6. Maintenance crew is service facility and repairing the machine is service.
7. Traffic signals are service facility and control of traffic is service.
8. Signal post is a service facility and green signaling is service.

Above we have seen elements or customer and service facility and service. We can see here that all the customer or elements (hereafter called as customer only) will arrive and waits to avail the service at service station. When the service station has no desired capacity to serve them all at a time the customer has to wait for his/its chance resulting the formulation of a waiting line of customers which is generally known as a queue. In general we can say that ***a flow of customers from infinite or finite population towards the service facility forms a queue or waiting line on account of lack of capability to serve them all at a time.*** The above discussion clarifies that the term customer we mean to the arriving unit that requires some service to be performed at the service station. **Queues** or **waiting lines** stands for a number of customers waiting to be serviced. Queue does not include the customer being serviced. The process or system that performs the services to the customer is termed as **service channel** or **service facility.**

Thus from the above we see that waiting lines or not only the lines formed by human beings butalso the other things like railway coaches, vehicles, material etc.

A.K.Erlang, a Danish telephone engineer, did original work on queuing theory. Erlang started his work in 1905 in an attempt to determine the effects of fluctuating service demand (arrivals) on the utilization of automatic dialing equipment. It has been only since the end of World War II thatwork on waiting line models has been extended to other kinds of problems. In today's scenario a wide variety of seemingly diverse problems situations are recognized as being described by the general waiting line model. In any queuing system, we have an input that arrives at some facility forservice or processing and the time between the arrivals of individual inputs at the service facility is commonly random in nature. Similarly, the time for service or processing is commonly a random variable.

Table 1. Waiting line model elements for some commonly known situations.

S.No.	*Situation.*	*Arriving element.*	*Service facility*	*Service or process*
1	Ship entering a port.	Ships	Docks	Unloading and loading.
2.	Maintenance and repair of machines	Machine break downs	Repair crew.	Repairing of machines.
3.	Non automatic assembly line	Parts to be assembled.	Individual assembly operations or entire line.	Assembly.
4.	Purchase of groceries at super market.	Customer with loaded grocerycarts.	Checkout Counter.	Tabulation of bill,receipt of paymentand bagging of groceries.
5.	Automobile and other vehicles at an intersection of roads.	Automobiles and vehicles.	Traffic signal lights.	Control of traffic.
6.	Inventory of itemsin stores or warehouse.	Order for withdrawal.	Store or warehouse.	Replenishment of inventory.
7.	Patients arriving atan hospital	Patients	Medical craw	Health care of the patient.

In figure number 2 arrows between service centers indicates possible routes for jobs processed in the shop. In this particular system, we see that the service center moves to the customer rather than the customer coming to service center for service. So, it may be understood here that there is no rule that always the customers has to move to service centers to get the service. Depending on the situation, the service center may also move to the customer to provide service. In this system the departure from one-service center may become input to the other service center.

In our everyday activity, we see that there is a flow of customer to avail some service from service facility. The rate of flow depends on the nature of service and the serving capacity of the station. In many situations there is a congestion of items arriving from service because an item cannot be serviced immediately on arrival and each new arrival has to wait for some time before it is attended. This situation occurs where the total number of customers requiring service exceeds the number of facilities. So we can define a queue as **“A group of customers / items waiting at some place to receive attention / service including those receiving the service.”**

In this situation, if queue length exceeds a limit, the customer get frustrated and leave the queue to get the service at some other service station.In this case the organization looses the customer goodwill.

Similarly some service facility waits for arrival of customers when the total capacity of system is more than the number of customers requiring service. In this case service facility remains idle for a considerable time causing a burden of exchequer.

So, in absence of a perfect balance between the service facility and the customers, waiting is required either by the customer or by the service facility. The imbalance between the customer and service facility, known as congestion, cannot be eliminated completely but efforts / techniques can be evolved and applied to reduce the magnitude of congestion or waiting time of a new arrival in the system or the service station. The method of reducing congestion by the expansion of servicing counter may result in an increase in idle time of the service station and may become uneconomical for the organization. Thus both the situation namely of unreasonable long queue or expansion of servicing counters are uneconomical to individual or managers of the system.

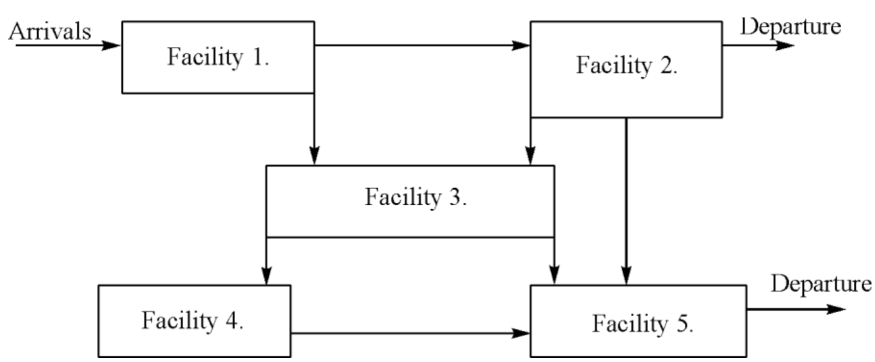


Figure 2. Complex queue for a maintenance shop.

As discussed above, if the length of the queue is longer, the waiting time of the customer will increase causing dissatisfaction of customer and to avoid the longer waiting time of customer, if the management increases the service facilities, then many a time we see that the service facilities will remain idle causing burden on the organization. To avoid this situation, the theory of waiting line will help us to reduce the waiting time of the customer and suggest the organization to install optimal number of service facilities, so that customer will be happy and the organization can run the business economically.

The arrival pattern of the customer and the service time of the facility depend on many factors and they are not under the control of the management. Both cannot be estimated or assessed in advance and moreover their arrival pattern and service time are random in nature. The waiting line phenomenon is the direct result of randomness in the operation of service facility and random arrival pattern of the customer. The customer arrival time cannot be known in advance to schedule the service time and the time required to serve each customer depends on the magnitude of the service required by the customer. For example, let us consider two customers who come to the ticket counter to purchase the counter. One-person tenders exact amount and purchase one ticket and leaves the queue. Another person purchases 10 tickets and gives a Rs. 500/- currency note. For him after giving the ticket, the counter clerk has to give the remaining amount back. So the time required for both customers will vary. The randomness of arrival pattern and service time makes the waiting line theory more complicated and needs careful study. The theory tries to strike a balance between the costs associated with waiting and costs of preventing waiting and help us to determine the optimal number of service facilities required and optimal arrival rate of the customers of the system.

HISTORICAL DEVELOPMENT OF THE THEORY

During 1903 Mr. A.K. Erlang, a Swedish engineer has started theoretical analysis of waiting line problem in telephone calls. In 1927, Mr. Millins developed the theory further and then by Mr. Thornton D Fry. But Mr. D.G.Kendall has given a systematic and mathematical approach to waiting line problem in 1951. After 1951 significant work has been done in waiting line theory, so as to enable it to apply to varieties of problems come across in industries and society. One best example of this may be quoted as the control of waiting time of a customer in queue complex of Tirupathi Temple. The present system of tying a belt with time to the hands of a customer is the results of application of queuing theory. Another example is computerized reservation of rail journey.

QUEUING SYSTEM OR PROCESS

One thing we have to remember is that when we speak of queue, we have to deal with two elements,

i.e. Arrivals and Service facility. Entire queuing system can be completely described by:

(a) The input (Arrival pattern)
(b) The service mechanism or service pattern,
(c) The queue discipline and
(d) Customer behavior.

Components of the queuing system are arrivals, the element waiting in the queue, the unit being served, the service facility and the unit leaving the queue after service.

QUEUING PROBLEMS

The most important information required to solve a waiting line problem is the nature and probability distribution of arrivals and service pattern. The answer to any waiting line problem depending on finding:

(a) *Queue length*: The probability distribution of queue length or the number of persons in the system at any point of time. Further we can estimate the probability that there is no queue.
(b) *Waiting time*: This is probability distribution of waiting time of customers in the

queue. That is we have to find the time spent by a customer in the queue before the commencement of his service, which is called *his waiting time in the queue*. The total time spent in the system is the waiting time in the queue plus the service time. The waiting time depends on various factors, such as:

(i) The number of units already waiting in the system,

(ii) The number of service stations in the system,

(iii) The schedule in which units are selected for service,

(iv) The nature and magnitude of service being given to the element being served.

(c) *Service time:* It is the time taken for serving a particular arrival.

Average idle time or Busy time distribution: The average time for which the system remains idle. We can estimate the probability distribution of busy periods. If we suppose that the server is idle initially and the customer arrives, he will be provided service immediately. During his service time some more customers will arrive and will be served in their turn according to the system discipline. This process will continue in this way until no customer is left unserved and the server becomes free again after serving all the customers. At this stage we can conclude, that the busy period is over. On the other hand, during the idle periods no customer is present in the system. A busy period and the idle period following it together constitute a *busy cycle*. The study of busy period is of great interest in cases where technical features of the server and its capacity for continuous operation must be taken into account.

STEADY, TRANSIENT AND EXPLOSIVE STATES IN A QUEUE SYSTEM

The distribution of customer's arrival time and service time are the two constituents, which constitutes of study of waiting line. Under a fixed condition of customer arrivals and service facility a queue length is a function of time. As such a queue system can be considered as some sort of random experiment and the various events of the experiment can be taken to be various changes occurring in the system at any time. We can identify three states of nature in case of arrivals in a queue system. They are named as ***steady state, transient state, and the explosive state.***

(a) *Steady State:* The system will settle down as steady state when the rate of arrivals of customers is less than the rate of service and both are constant. The system not only becomes steady state but also becomes independent of the initial state of the queue. Then the probability of finding a particular length of the queue at any time will be same. Though the size of the queue fluctuates in steady state the statistical behaviour of the queue remains steady. Hence we can say that *a steady state condition is said to prevail when the behaviour of the system becomes independent of time.*

A necessary condition for the steady state to be reached is that elapsed time since the start of the operation becomes sufficiently large *i.e.* ($t \to \infty$), but this condition is not sufficient as the existence of steady state also depend upon the behaviour of the system *i.e.* if the rate of arrival is greater than the rate of service then a steady state cannot be reached. Hence we assume here that the system acquires a steady state as $t \to \infty$ *i.e.* the number of arrivals during a certain interval becomes independent of time. *i.e.*

$$\lim_{t \to \infty} P_n(t) \to P_n$$

Hence in the *steady state system, the probability distribution of arrivals, waiting time, and service time does not depend on time.*

(*b*) *Transient State*

Queuing theory analysis involves the study of a system's behaviour over time. A system is said to be in 'transient state' when its operating characteristics or behaviour are dependent on time. This happens usually at initial stages of operation of the system, where its behaviour is still dependent on the initial conditions. So when the probability distribution of arrivals, waiting time and servicing time are dependent on time the system is said to be in transient state.

(*b*) *Explosive State*

In a situation, where arrival rate of the system is larger than its service rate, a steady state cannot be reached regardless of the length of the elapsed time. Here queue length will increase with time and theoretically it could build up to infinity. Such case is called the *explosive state*.

In our further discussion, all the problems and situations are dealt with steady state only.

DESIGNATION OF QUEUE AND SYMBOLS USED IN QUEUING MODELS

A queue is designated or described as shown below: A model is expressed as
A/B/S : (*d* / *f*) where,

A: Arrival pattern of the units, given by the probability distribution of inter - arrival time of units. For example, Poisson distribution, Erlang distribution, and inter arrival time is 1 minute or 10 units arrive in 30 minutes etc.

B: The probability distribution of service time of individual being actually served. For example the service time follows negative exponential distribution and 10 units are served in 10 minutes or the service time is 3 minutes, etc.

S: The number of service channels in the system. For example the item is served at one service facility or the person will receive service at 3 facilities etc.

d: Capacity of the system. That is the maximum number of units the system can accommodate at any time. For example, the system has limited capacity of 40 units or the system has infinite capacity etc.

f: The manner or order in which the arriving units are taken into service *i.e*. FIFO / LIFO / SIRO /Priority.

NOTATIONS

X: Inter arrival time between two successive customers (arrivals).

Y: The service time required by any customer.

w: The waiting time for any customer before it is taken into service.

v: Time spent by the customer in the system.

n: Number of customers in the system, that is customers in the waiting line at any time, including the number of customers being served.

P_n (t): Probability that '*n*' customers arrive in the system in time '*t*'.

□$_n$ **(t):** Probability that '*n*' units are served in time '*t*'.

U (T): Probability distribution of inter arrival time ***P* (*t* □ *T*).V (T):** Probability distribution of servicing time ***P* (*t* □ *T*).**

F (N): Probability distribution of queue length at any time ***P* (*N* □ *n*).**

E_n: Some state of the system at a time when there are '*n*' units in the system.

□$_n$: Average number of customers arriving per unit of time, when there are already '*n*' units in the system.

□ : Average number of customers arriving per unit of time.

□$_n$: Average number of customers being served per unit of time when there are already '*n*' unitsin the system.

□ **:** Average number of customers being served per unit of time.

1 / □ : Inter arrival time between two arrivals.

1 /□ : Service time between two units or customers.

ρ = (□ / □) :System utility or traffic intensity which tells us how much time the system was utilized in a given time. For example given time is 8 hours and if ρ = 3 / 8, it means to say that out of 8 hours the system is used for 3 hours and (8 – 3 = 5) 5 hours the is idle.

QUEUE MODELS

Most elementary queuing models assume that the *inputs / arrivals* and *outputs / departures* follow a **birth** and **death** process. Any queuing model is characterized by situations where both arrivals and departures take place simultaneously. Depending upon the nature of inputs and service faculties, there can be a number of queuing models as shown below:

(i) **Probabilistic queuing model:** Both arrival and service rates are some unknown random variables.

(ii) **Deterministic queuing model:** Both arrival and service rates are known and fixed.

(iii) **Mixed queuing model:** Either of the arrival and service rates is unknown random variable and other known and fixed.

Earlier we have seen how to designate a queue. Arrival pattern / Service pattern / Number of channels / (Capacity / Order of servicing). (*A* /*B*/ *S* / (*d* / *f*).

In general ***M*** is used to denote Poisson distribution (**Markovian**) of arrivals and departures.

D is used to constant or Deterministic distribution.

E_k is used to represent Erlangian probability distribution.

G is used to show some general probability distribution.

In general queuing models are used to explain the descriptive behavior of a queuing system. These quantify the effect of decision variables on the expected waiting times and waiting lengths as well as generate waiting cost and service cost information. The various systems can be evaluated through these aspects and the system, which offers the minimum total cost is selected.

Procedure for Solution

(a) List the alternative queuing system

(b) Evaluate the system in terms of various times, length and costs.

(c) Select the best queuing system.

(Note: Students / readers are advised to refer to the books on Operations Research written with mathematical orientation for the derivation of formulas for various queuing models. In this book, the application of formula is made.)

Poisson Arrival / Poisson output / Number of channels / Infinite capacity /FIFO Model

M / M / 1 / (∞ / FIFO):

Formulae used

1. Average number of arrivals per unit of time = λ
2. Average number of units served per unit of time = μ
3. Traffic intensity or utility ratio = $\rho = \frac{\lambda}{\mu}$ the condition is : $(\lambda < \mu)$
4. Probability that the system is empty = $P_0 = (1 - \rho)$
5. Probability that there are 'n' units in the system = $P_n = \rho^n P_0$
6. Average number of units in the system = $E(n) = \frac{\rho}{(1-\rho)}$ or $= \frac{\lambda}{(\mu-\lambda)} = L_q + \frac{\lambda}{\mu}$
7. Average number of units in the waiting line = $E_L = \frac{\rho^2}{(1-\rho)}$ $\frac{\lambda^2}{\mu(\mu-\lambda)}$
8. Average waiting length (mean time in the system) = $E(L / L > 0)$

$$= \frac{1}{(\mu-\lambda)} \quad \frac{1}{(1-\rho)}$$

$$= E(w) + \frac{1}{\mu} = \frac{L}{\lambda}$$

9. Average length of waiting line with the condition that it is always greater than zero

$$= V(n) = \frac{\rho}{(1-\rho)^2} = \frac{\lambda\mu}{(\mu-\lambda)^2}$$

$$= \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu-\lambda)}$$

11. Average time an arrival spends in the system = $E(v)$ = $\frac{1}{\mu(1-\rho)} = \frac{1}{(\mu-\lambda)} = E(w/w > 0)$

12. P (w > 0) = System is busy = ρ

13. Idle time = $(1 - \rho)$

14. Probability distibution of waiting time = $P(w)\,dw$ =

$\mu\rho(1-\rho)\,e^{-\mu w(1-\rho)}$

15. Probability that a consumer has to wait on arrival = (P (w > 0) = ρ

16. Probability that a new arrival stays in the system =

$P(v)\,dv = \mu(1-\rho)\,e^{-\mu v(1-\rho)}\,dv,$

Problem 1.

A T.V. Repairman finds that the time spent on his jobs have an exponential distribution with mean of 30 minutes. If he repairs sets in the order in which they come in, and if the arrival of sets is approximately Poisson with an average rate of 10 per 8 hour day, what is repairman's expected idle time each day? How many jobs are ahead of the average set just brought in?

Solution

This problem is Poisson arrival/Negative exponential service / single channel /infinite capacity/ FIFO type problem.

Data: λ = 10 sets per 8 hour day = 10 / 8 = 5/4 sets per hour.

Given $1/\mu$ = 30 minutes, hence μ = (1/30) × 60 = 2 sets per hour.

Hence, Utility ratio = ρ = (λ/μ) = (5/4) / 2 = = 5 / 8. = 0.625. This means out of 8 hours 5 hours the system is busy *i.e.* repairman is busy.

Probability that there is no queue = The system is idle = $(1 - \rho)$ = 1 – (5 / 8) = 3 / 8 = That is outof 8 hours the repairman will be idle for 3 hours.

Number of sets ahead of the set just entered = Average number of sets in system = λ /

(□ □ □) =

$= \rho / (1 - \rho) = 0.625 / (1 - 0.625) = 5 / 3$ ahead of jobs just came in.

Problem 2.

The arrivals at a telephone booth are considered to be following Poisson law of distribution with an average time of 10 minutes between one arrival and the next. Length of the phone call is assumed to be distributed exponentially with a mean of 3 minutes.

(a) What is the probability that a person arriving at the booth will have to wait?

(b) What is the average length of queue that forms from time to time?

(c) The telephone department will install a second booth when convinced that an arrival would expect to wait at least thee minutes for the phone. By how much must the flow of arrivals be increased in order to justify a second booth?

Solution

Data: Time interval between two arrivals = 10 min. = $1/\lambda$, Length of phone call = 3 min. = $1/\mu$.

Hence $\lambda = 1/10 = 0.1$– per min and $\mu = 1/3 = 0.33$ per min., and $\rho = \lambda/\mu = 0.10 / 0.33 = 0.3$

(a) Any person who is coming to booth has to wait when there is somebody in the queue. He need not wait when there is nobody in the queue *i.e.* the queue is empty. Hence the probabilityof that an arrival does not wait = $P_0 = (1 - \rho)$.

Hence the probability that an arrival has to wait = 1 – The probability that an arrival does not wait = $(1 - P_0) = 1 - (1 - \rho) = \rho = 0.3$. That means 30% of the time the fresh arrival has to wait. That means that 70 % of the time the system is idle.

(b) Average length of non- empty queue from time to time = **(Average length of the waiting line with the condition that it is always greater than zero =** ***i.e.*** $E(L/L > 0)$**)** $1(1 - \rho)$

$= 1 / (1 - 0.3) = 1.43$ persons.

(c) The installation of the second booth is justified if the waiting time is greater than or equal tothree. If the new arrival rate is λ', then for $\mu = 0.33$ we can work out the length of the waiting line. In this case $\rho = \lambda'/\mu$.

Length of the waiting line for λ' and $\mu = 0.33 = E(w) = \{\lambda'/\mu(\mu - \lambda')\} \geq 3$ or $\lambda' \geq (3\mu^2 - 3\mu\lambda')$ or $\lambda' \geq (3\mu^2)/(1 + 3\mu) = (3 \times 0.33^2) / (1 + 3 \times 0.33)$ *i.e.* $\lambda' \geq 0.16$. That is the arrival rate must be at least 0.16 persons per minute or one arrival in every 6 minutes. This can be written as 10 arrivals per hour to justify the second booth.

Problem 3.

In a departmental store one cashier is there to serve the customers. And the customers pick up their needs by themselves. The arrival rate is 9 customers for every 5 minutes and the cashier can serve 10 customers in 5 minutes. Assuming Poisson arrival rate and exponential distribution for service rate,find:

(a) Average number of customers in the system.
(b) Average number of customers in the queue or average queue length.
(c) Average time a customer spends in the system.
(d) Average time a customer waits before being served.

Solution

Data: Arrival rate is □ = (9 / 5) = 1.8 customers per minute.

Service rate = □ = (10 / 5) = 2 customers per minute. Hence ρ □ (□ / □) □ (1.8 / 2) = 0.9

(a) Average number of customers in the system = $E(n)$ = ρ / (1 □ ρ) = 0.9 / (1 – 0.9)
= 0.9 / 0.1
= 9 customers.

(b) Average time a customer spends in the system = $E(v)$ = 1/ □ (1 □ ρ) □ 1/(□ □ □) = 1 / (2 – 1.8) = 5 minutes.

(c) Average number of customers in the queue = $E(L)$
= □^{2} /(1 □ ρ) □ □^{2} / □(□ □ □) □ (ρ) □ □ /(□ □ □) = 0.9 × 1.8 / (2 – 1.8) = 8.1 customers.

(d) Average time a customer spends in the queue = ρ / □ (1 □ ρ) □ □ / □(□ □ □) = 0.9 / 2 (1 – 0.9)
= 0.9 / 0.2 =4.5 minutes.

Problem 4.

A branch of a Nationalized bank has only one typist. Since typing work varies in length (number of pages to be typed), the typing rate is randomly distributed approximating a Poisson distribution with a mean service rate of 8 letters per hour. The letter arrives at a rate of 5 per hour during the entire 8- hour workday. If the typist is valued at Rs. 1.50 per hour, determine:

(*a*) Equipment utilization, (*b*) The percent time an arriving letter has to wait, (*c*) Average system time, and d) Average idle time cost of the typewriter per day.

Solution

Data = arrival rate = □ = 5, Service rate □ = 8 per hour.

Hence ρ □ (□ / □) = 5 / 8 = 0.625

(a) Equipment utilization = Utility ratio = ρ = 0.625, *i.e.* 62.5 percent of 8 hour day the equipmentis engaged.

(b) Percent time that an arriving letter has to wait = As the machine is busy for 62.5 % of the day, the arriving letter has to wait for 62.5 % of the time.

(c) Average system time = Expected (average) a customer spends in the system = 1/(□ □ □) = [1 / (8 – 5)] = 1/3 hour. = 20 minutes.

d) Average idle time cost of the typewriter per day = 8 hours × idle time × idle time cost =
= 8 × (1 – 5 / 8) × Rs. 1.50 = Rs. 4.50.

Problem 5.

A product manufacturing plant at a city distributes its products by trucks, loaded at the factory warehouse. It has its own fleet of trucks plus trucks of a private transport company. This transport company has complained that sometimes its trucks have to wait in line and thus the company loses money paid for a truck and driver of waiting truck. The company has asked the plant manager either to go in for a second warehouse or discount prices equivalent to the waiting time. The data available is:

Average arrival rate of all trucks = 3 per hour. Average service rate is = 4 per hour.

The transport company has provided 40% of the total number of trucks. Assuming that theserates are random according to Poisson distribution, determine:

(a) The probability that a truck has to wait?

(b) The waiting time of a truck that has to wait,

(c) The expected waiting time of company trucks per day.

Solution

Data: □ = 3 trucks per hour, □ = 4 trucks per hour. Hence ρ = utilization factor = (□ / □) = 3 / 4 = 0.75. This means that the system is utilized 75% of the time. Hence 75% the time the truck has towait.

The waiting time of truck that waits = E (v) = 1/(□ □ □) = 1 / (4 – 3) = 1 hour.

Total expected waiting time of company trucks per day = (Trucks per day) × (% company trucks) × Expected waiting time per truck. = (3 × 8) × (0.40) × [□ / □(□ □ □)] = 24 × 0.40 × [3 / 4(4- 3) = 24 × 0.40 × 0.75 = 7.2 hours per day.

UNIT – V REPLACEMENT ANALYSIS AND SIMULATION

INTRODUCTION

The problem of replacement arises when any one of the components of productive resources, such as machinery, building and men deteriorates due to time or usage. The examples are:

(a) A machine, which is purchased and installed in a production system, due to usage some of its components wear out and its efficiency is reduced.

(b) A building in which production activities are carried out, may leave cracks in walls, roof etc, and needs repair.

(c) A worker, when he is young, will work efficiently, as the time passes becomes old and his work efficiency falls down and after some time he will become unable to work.

In general, when any production facility is new, it works with full operating efficiency and due to usage or of time, it may become old and some of its components wear out and the operating efficiency of the facility falls down. To regain the efficiency, a remedy by, namely ***maintenance*** is to be attended. The act of maintenance consists of replacing the worn out part, or oiling or overhauling, or repair etc. In modern industrial scene, the presence of highly sophisticated machinery in manufacturing system will bother the manager, when any one of the facilities goes out of order or breakdown. Because of the breakdown of one of the facility, the entire production system may be affected. This is particularly true in batch manufacturing system and continuous manufacturing system. The loss of production hours is more expensive factor for the manufacturing unit. Hence, the management must take interest in maintaing the production facility properly, so that facility's available time will be more than the down time. All the production facilities are subjected to deterioration due to their use and exposure to the environmental conditions. The process of deterioration, if unchecked or neglected, it culminates and makes the facility useless. Hence, the management has to check the facilities periodically, and keep all the facilities in operating condition. Once the maintenance is attended, the efficiency may not be regained to previous level but a bit less than that of previous level. For example, if the operating efficiency is 95 percent and due to deterioration, the efficiency reduces to 90 percent, after maintenance, it may regain to the level of 93 percent. Once again due to usage the efficiency falls down and the maintenance is to be attended. This is an ongoing business of the management. After some time, the efficiency reduces to such a level, the maintenance cost will become very high and due to less efficiency the unit production cost will be very high and this is the time the management has to think of replacing the facility. This may be well explained by means of a figure. Referring to figure 1, the operating efficiency at the beginning is 95%. When first maintenance is attended, it is reduced to 93%. In the second maintenance it is reduced to 80 percent. Like this the facility deteriorates, and finally the operating efficiency reduces to 50 percent, where the it is not economical to use the facility for further production, as the maintenance cost will be very high, and the unit production cost also increases, hence the replacement of the facility is due at this stage. In this chapter, we will discuss the mathematical models used for finding the optimal replacement time of facilities.

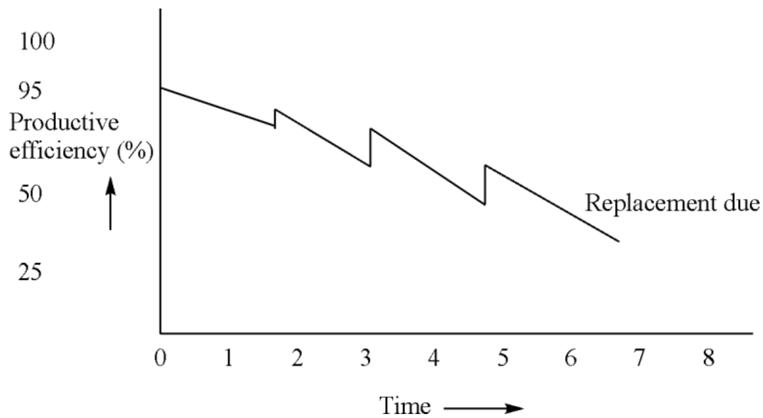


Fig. 1

Thus the problem of replacement is experienced in systems where machines, individuals or capital assets are the main production or job performing units. The characteristics of these units is that their level of performance or efficiency decreases with time or usage and one has to formulate some suitable replacement policy regarding these units to keep the system up to some desired level of performance. We may have to take different type of decision such as:

(a) We may decide whether to wait for complete failure of the item (which may result in some losses due to deterioration or to replace earlier at the expense of higher cost of the item,

(b) The expensive item may be considered individually to decide whether we should replace now or, if not, when it should be reconsidered for replacement,

(c) Whether the item is to be replaced by similar type of item or by different type for example item with latest technology

The problem of replacement is encountered in the case of both men and machines. Using probability,it is possible to estimate the chance of death or failure at various ages. The main objective of replacement is to help the organization for maximizing its profit or to minimize the cost.

FAILURE MECHANISIM OF ITEMS

The word failure has got a wider meaning in **industrial maintenance** than what it has in our daily life. We can categorize the failure in two classes. They are (*i*) ***Gradual failure*** and (*ii*) ***Sudden failure.*** Once again the sudden failure may be classified as: (*a*) ***Progressive failure,*** (*b*) ***Retrogressive failure*** and (*c*) ***Random failure***.

(i) Gradual failure:

In this class as the life of the machine increases or due continuous usage, due to wear and tear of components of the facility, its efficiency deteriorates due to which the management can experience:

(*a*) Progressive Increase in maintenance expenditure or operating costs, (*b*) Decreased productivity of the equipment and (*c*) decrease in the value of the equipment *i.e.* resale value of the equipment/facility decreases.

Examples of this category are: Automobiles, Machine tools, etc.

(ii) ***Sudden failure:***

In this case, the items ultimately fail after a period of time. The life of the equipment cannot be predicted and is some sort of random variable. The period between installation and failure is not constant for any particular type of equipment but will follow some frequency distribution, which may be:

(a) **Progressive failure:** In this case probability of failure increases with the increase in life of an item. The best example is electrical bulbs and computer components. It can be shown as in figure 2 (*a*).

(b) **Retrogressive failure:** Some items will have higher probability of failure in the beginning of their life, and as the time passes chances of failure becomes less. That is the ability of the item to survive in the initial period of life increases its expected life. The examples are newly installed machines in production systems, new vehicles, and infant baby (The probability of survival is very less in infant age, but once the baby get accustomed to nature, the probability of failure decreases). This can be shown as in figure 2 (*b*).

(c) **Random failure:** In this class, constant probability of failure is associated with items that fail from random causes such as physical shocks, not related to age. In such cases all items fail before aging has any effect. This can be shown as in figure 2. (*c*). Example is vacuum tubes.

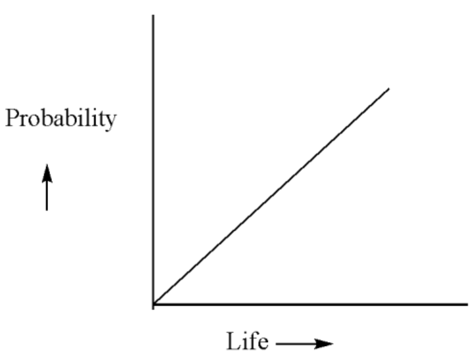


Figure 2(a) Progressive failure
Probability of failure increases with life of the item

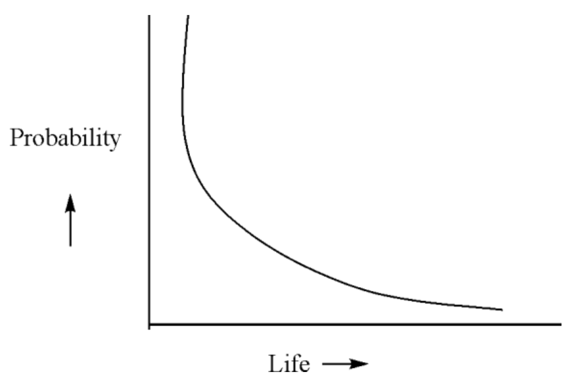


Figure 2(b)
Retrogressive failure
Probability of failure is more in the nore in early life of the item and then chance of failure decreases.

Figure 7.3

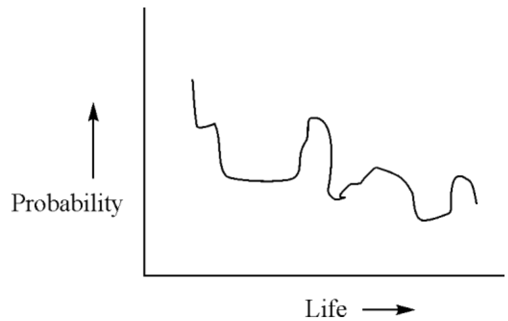


Figure 2(c) Random failure

Item fail due to some random cause but not due to ageThe above may be shown as in figure 3

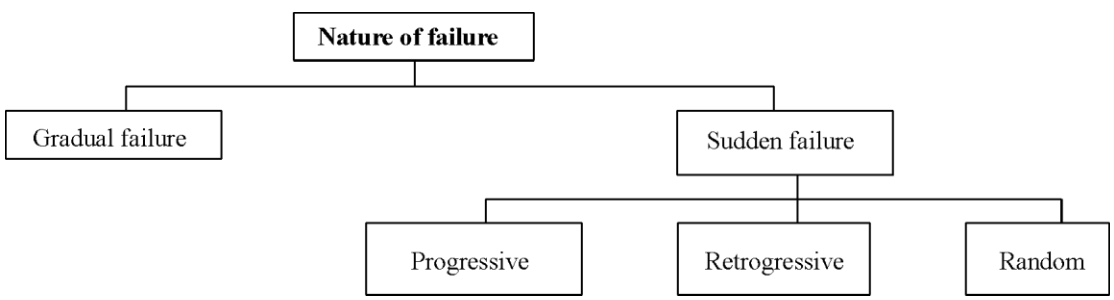


Figure 3

Bathtub Curve

Machine or equipment or facility life can be classified into three stages. They are ***Infant stage, Youth stage or Youth phase and Old age stage or Old age phase.***

(a) Infant stage or phase

This is also known as early failure stage. When new equipment is purchased and installed in the existing system, it has to cope up with the operating efficiency of the existing system. Also it has to accustom to operating skill of the operator. Perhaps when a new vehicle is purchased, due to mechanical conditions of the machine and the operating skill of the person, the vehicle may give trouble in the early stages. The owner may have to visit the mechanic many times. It is like a baby, which has come out from mother's womb. The baby until birth, it was in a controlled atmosphere and when the birth takes place, it has to get accustom to outside atmosphere. Hence it cries. Sometimes there is a danger of failure of life. Hence we see more failures in infant stage. Once an age of 10 to

15 years is reached, the death rate or failure rates will reduce. In this stage the safety of machine is covered by the guarantee period given by the manufacturer. This is represented by a curve on the left hand side of the **Bathtub curve.**

(b) Youth stage or random failure stage

In this stage the equipment or machine is accustomed to the system in which it is installed and works at designed operating efficiency. Regular maintenance such as overhauling, oiling, greasing, cleaning keep the machine working. Now and then due to wear and tear of components or heavy load or electrical voltage fluctuations, breakdown may occur, which can be taken care of by repair maintenance. The machine or equipment works for longer periods without any trouble. This is like youth stage in human life that is full of vigor and energy and the person will be healthy and work for longer periods without any diseases. This is shown as a horizontal line in bathtub curve. Here repair maintenance; preventive maintenance or other maintenance techniques are used to keep the machine or equipment in working condition.

(c) Old age stage or Old age phase or Wear out failures

Due to continuous usage and age of the machine, there will be wear and tear of various components. Not only this, during youth stage, some of the components might have been replaced due to wear and tear. These replaced components may not suit well in the system if they are not from original manufacturer. As the manufacturers are changing the design, one has to go for spares available in the market. All this may reduce to operating efficiency of the machine or equipment and the management has to face frequent failures. This is very similar to old age in human life. Due to old age, people will get diseases and old age weakness and many a time they have to go to hospitals for treatment before the life fails. This is shown on the right side of the bath- tub curve. Here one will think of replacement of the equipment or machine. When all the above three curves are assembled, we get a curve which is in the form of a **bathtub** and is known as **bathtub curve,** figure 4.

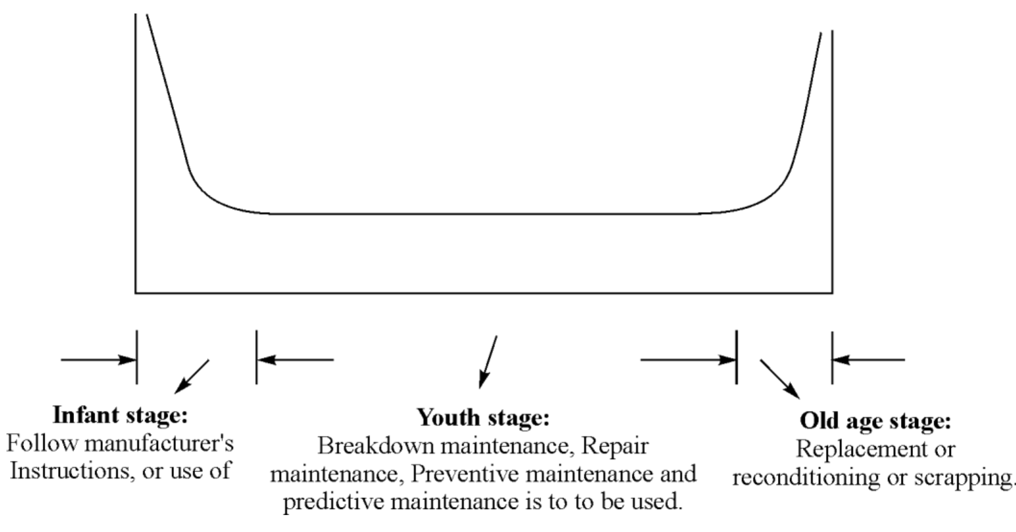


Figure 4 Bathtub curve.

All the above-discussed points may be summarized as:

Table 1. Summary of three- stages of maintenance

Phase	Type of Failure	Failure Rate	Causes of failure	Cost of Failure	Suitable maintenance Policy
Infant phase	Early Failures	Decreeing Trend.	Faulty design, Erratic Operation, Installation Errors, environmental Problems.	Medium to high	Warrantee / guarantee by manufacturer.
Youth phase	Random or Chanced Failures or Rare event Failures.	Constant	Operational errors, Heavy load, over run	Low to medium	Break down, Predictive Preventive, Repair Maintenance etc.
Old age phase	Wear out or Age failures due to wear and tear..	Increasing	Wear, tear, Creep, Fatigue etc.	Low High	Operate to fail and Corrective maintenance Reconditioning or Replacement.

Costs Associated with Maintenance

Our main aim in this chapter is to find optimal replacement period so as to minimize the maintenance cost. Hence we are very much interested in the various cost associated with maintenance. Various costs to be discussed are:

(a) Purchase cost or Capital cost: (C)

This cost is independent of the age of the machine or usage of the machine. This is incurred at the beginning of the life of the machine, *i.e.* at the time of purchasing the machine or equipment. But the interest on the invested money is an important factor to be considered.

(b) Salvage value / Scrap value / Resale value / Depreciation: (S)

As the age of the machine increases, the resale value decreases as its operating efficiency decreases and the maintenance costs increases. It depends on the operating conditions of the machine and life of the machine.

(c) Running costs including maintenance, Repair and Operating costs:

These costs are the functions of age of the machine and usage of the machine. As the usage increases or the age increases, due to wear and tear, many components fail to work and they are to be replaced. As the age increases, failures also increase and the maintenance costs goes on increasing. At some period the maintenance costs are so high, which will indicate that the replacement of the machine or equipment is essential.

These costs can be shown by means of a curve as in figure

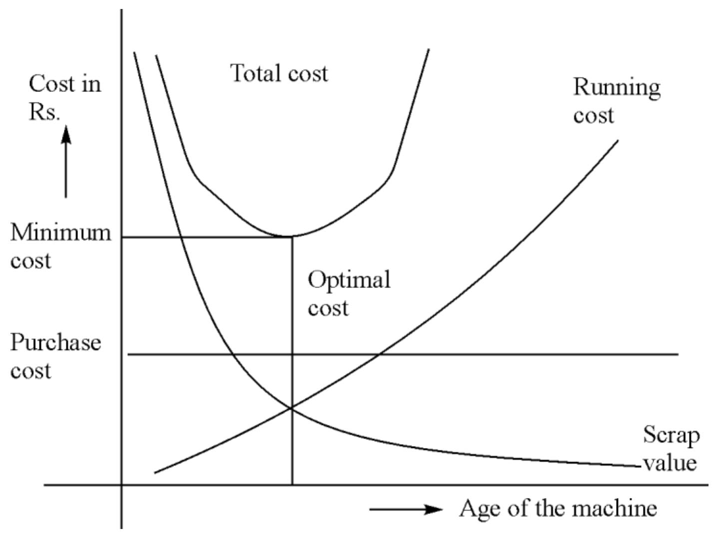


Figure 5.

TYPES OF REPLACEMET PROBLEMS

One must remember that the study of **Replacement of items** is a field of application rather than a method of analysis. The study involves, the comparison of alternative replacement policies. Various types of replacement problems we come across in this chapter is:

(a) Replacement of **Capital equipment,** which looses its operating efficiency due to aging (passage of time), or due to continuous usage (due to wear and tear of components). Examples are: Machine tools, Transport and other vehicles, etc., Here the system can maintain the level of performance by installing a new unit at the beginning of some unit of time (year, month or week) and decide to keep it up to some suitable period so as to minimize the operating and maintenance costs. In this case the deterioration process is predictable and is represented by an increased maintenance cost and decreased in scrap cost and increased production cost per unit. In such cases the optimum life of the item is determined on the assumption that increased age reduces efficiency. Deterministic models explain the problem and they are very much similar to that of inventory models where deterioration corresponds to demand against the desired level of efficiency (level of inventory). The cost of new item is similar to cost of replenishment of inventory and maintenance cost corresponds to cost ofholding inventory.

These types of problems are solved by two methods. They are:

(i) By calculating the cost per unit of time, without considering the money value. Here we calculate the total cost up to the period and divide by time unit (years, months, weeks etc.,) to find the average cost to decide the period of replacement.

(ii) By taking the money value into consideration using ***present value concept*** to compare on a one number basis.

(a) Replacement of **items that fail completely** all in a sudden in a random nature. We use **Group replacement or Preventive maintenance technique** for these items and these are expensive to replace individually. Examples are: Electric bulbs, Transistors, Electronic components etc., Here replacement of items are done in anticipation of failure, which is known as ***preventive maintenance.*** We assume that the items will have relatively constant efficiency until they fail or die. These models require the knowledge of statistics and stochastic process involving probability of failure. The replacement policy is formulated to balance the wasted life of items replaced before failure against the costs incurred when items fail in service.

(b) Replacement of **human beings in organizations,** known as **Staffing problem,** or known as **Human resource planning** or **Mortality and Staffing problem.** This problem requires the knowledge of life distribution for service of staff in a system.

(c) Miscellaneous problems such as replacement of existing units due to availability of more effective and new and advanced technology. In these problems replacement will become necessary due to research of new and advanced and more effective technology and old technology becomes out of date.

GENERAL APPROACH TO SOLUTION TO REPLACEMENT PROBLEM

Though it is not possible or it is difficult to predict the time of failure of an item exactly, likely failure pattern could be established by observation. Generating the probability distribution for the given situation and then using them in conjunction with relevant cost information we can formulate the optimum replacement policy. The information necessary to formulate optimum replacement policy is:

(i) Objective assessment of the probability of the item failing at a particular point of time

(ii) Assessments of the cost of replacement in terms of:

  (a) Actual cost of the item,

  (b) Direct costs of labour involved in replacement,

  (c) Costs of disruption in terms of lost production, lost orders etc.,

REPLACEMENT OF ITEMS WHOSE EFFICIENCY REDUCES OR MAINTENCNCE COST INCREASES WITH TIME OR DUE TO AGE AND MONEYVALUE IS NOT CONSIDERED

Costs to be considered: Various cost items to be considered in replacement decisions are the costs that depend upon the choice or age of item or equipment. The costs those do not change with the age of the machine or item need not be taken into consideration. The replacement of items whose efficiency reduces with time is justified when the average cost per time period goes on reducing longer the replacement is postponed. However, there will come an age at which the rate of increase of running costs more than compensates the saving in average capital costs. At this age the replacement is justified. In the case of replacement of items whose efficiency deteriorates with time, the most important criteria to be considered is the measurement of efficiency. Consider a machine, in this case, the maintenance cost always increases with time and usage and a time comes when the maintenance cost becomes large enough, which indicates that it is better and economical to replace the machine with a new one. When we want to replace the machine, we may come across various alternative choices, where we have to compare the various cost elements such as running costs and maintenance costs to select optimal choice. The various techniques we may come across to analyze the situation are:

(a) Replacement of items whose maintenance cost increases with time and ***value of moneyremains same*** during the period,

(b) Replacement of items whose maintenance cost increases with time and the ***value of money also changes*** with time, and

(c) To compare alternative choices, use of ***concept of present value.***

Problem 1.

A firm is thinking of replacing a particular machine whose cost price is Rs. 12,200. The scrap value of the machine is Rs. 200/-. The maintenance costs are found to be as follows:

Year	1	2	3	4	5	6	7	8
Maintenance Cost in Rs.	220	500	800	1200	1800	2500	3200	4000

Determine when the firm should get the machine replaced.

Solution

Year (t)Y	*u (t) Maintenance Cost. (Rs)*	*M (y) = y □ u (t)t = 1*	*C = Capital Cost in Rs.*	*Scrap Cost (S) In Rs.*	*T (y) = C - S + M (y)*	*Average Cost : G (y) = T (y) / y*
1	2	3	4	5	6	7
1	2	34		5	6 = 4-5 +3	7 = 6/1
1	220	220	12200	200	12220	12220
2	500	720	12200	200	12720	6360
3	800	1520	12200	200	13520	4506.67
4	1200	2720	12200	200	14720	3680
5	1800	4520	12200	200	16520	3304
6	2500	7020	12200	200	19020	**3170**
7	3200	10220	12200	200	22220	3174.29
8	4000	14220	12200	200	26220	3277.5

Replace the machine at the end of 6th year when the average annual maintenance cost is minimum.

Problem 2.

The initial cost of a machine is Rs. 6100/- and its scrap value is Rs.100/-. The maintenance costs found from experience are as follows:

Year:	1	2	3	4	5	6	7	8
Annual maintenance cost in Rs.:	100	250	400	600	900	1200	1600	2000

When should the machine be replaced?

Solution

The time period is discrete. We have to find the period when the average maintenance cost will be minimum.

Years 't' = y	*u (t) Rs*	*$M(y)=\sum_{t=1}^{y} u(t)$*	*T (y) = C-S + M (y) Rs.*	*G (y) = T (y) / y Rs.*
1	100	100	6100	6100/1 = 6100
2	250	350	6350	6350 / 2 = 3175
3	400	750	6750	6750 / 3 = 2250
4	600	1350	7350	7350 / 4 = 1837.50
5	900	2250	8250	8250 / 5 = 1650
6	1200	3450	9450	**6450 / 6 = 1575**
7	1600	5050	11050	11050 / 7 = 1578.57
8	200	7050	13050	13050 / 8 = 1631.25

The annual average maintenance cost is minimum at the end of 6th year and it goes on increasing from 7th year. Hence the machine is to be replaced at the end of 6th year.

Problem 3.

The maintenance cost and resale value per year of a machine whose purchase price is Rs. 7000/
- is given below:

Year:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Maintenance cost in Rs.:	900	1200	1600	2100	2800	3700	4700	5900
Resale value in Rs.:	4000	2000	1200	600	500	400	400	400

When should the machine be replaced?

Solution

Years (t) = y	*Running Cost u (y)In Rs.*	*Cumulative Running costM (Y) in Rs.*	*Resale Value S (y)In Rs.*	*C - S (y) In Rs. C = 7000/-*	*T (y) = C – S (y) + M (y)In Rs.*	*T (y) / y = G (y) Average cost in Rs.*
1	900	900	4000	3000	3900	3900
2	1200	2100	2000	5000	7100	3550
3	1600	3700	1200	5800	9500	3166.67
4	2100	5800	600	6400	12200	3050
5	2800	8600	500	6500	15100	**3020**
6	3700	12300	400	6600	18900	3150
7	4700	17000	400	6600	23600	3371.43

8	5900	2 2900	400	6600	29500	3687.50

From the table we can see that the average cost is minimum at the end of the 5th year. Hence the machine may be replaced at the end of the 5th year.

Problem 4

A fleet owner finds form his past records that the cost per year of running a truck and resale values whose purchase price is Rs. 6000/- are given as under. At what stage the replacement is due?

Year:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Running cost in Rs.	1000	1200	1400	1800	230 0	280 0	340 0	4000
Resale value in Rs.	3000	1500	750	375	200	200	200	200

Solution

Let C = Capital cost = Rs. 6000/-, S (y) = Scrap value changes yearly, G (y) Average yearly cost,
u (t) = Annual maintenance cost.

Years (t) = y	*Running Cost u (y)In Rs.*	*Comulative Running costM (Y) in Rs.*	*Resale Value S (y)In Rs.*	*C - S (y) In Rs.*	*T (y) = C - S (y) + In Rs.*	*T (y) / y = G (y)Average cost inRs.*
1	1000	1000	3000	3000	4000	4000
2	1200	2200	1500	4500	6700	3350
3	1400	3600	750	5250	8850	2950
4	1800	5400	375	5625	11025	2756
5	2300	7700	200	5800	13500	**2700**
6	2800	10500	200	5800	16300	2717
7	3400	13900	200	5800	19700	2814
8	4000	17900	200	5800	23700	2962

From the table above we can see that the G (y) is minimum at the end of 5th year. Hence the truck is to be replaced at the end of 5th year.

Problem 5.

The initial cost of a vehicle is Rs. 3,800/- and the trade in value drops as time passes until it reaches Rs. 600/-. The maintenance costs are as shown below. Find when the replacement is due?

Year of service:	*1*	*2*	*3*	*4*	*5*
Year-end trade - in value in Rs.:	2000	120 0	800	700	600

Annual operating cost in Rs.:	1600	190 0	2200	2500	280 0
Annual Maintenance cost in Rs.	400	500	700	900	110 0

Solution

C = Capital cost in Rs. = 3800/-, *S* (*y*) =, Scrap value or trade in value, *u* (*t*) = Annual maintenance cost in Rs.

Here we can add operating cost and annual maintenance cost and put it together.

Years (t) = y	*Running Cost u (y)In Rs.*	*Cumulative Running costM (Y) In Rs.*	*Resale Value S (y)In Rs.*	*C – S (y) In Rs.*	*T (y) = C – S (y) + M (y)In Rs.*	*T (y) / y = G (y) Average costin Rs.*
1	200 0	2000	20 00	1800	3800	3800
2	240 0	4400	1200	2600	7000	3500
3	290 0	7300	800	3000	10300	**3433**
4	340 0	10700	700	3100	13800	3450
5	390 0	14600	600	3200	17800	3560

The optimal replacement period is at the end of third year. And the minimum annual average cost is Rs. 3433/-

Problem 6.

A Plant manager is considering the replacement policy for a new machine. He estimates the following costs in Rupees. Find an optimal replacement policy and corresponding minimum cost.

Year:	*1*	*2*	*3*	*4*	*5*	*6*
Replacement cost at the beginning of the year. (Rs)	100	110	125	140	160	190
Salvage value at the end of the year: (Rs)	60	50	40	25	10	0
Operating costs (Rs.)	25	30	40	50	65	80

Solution

In this problem Replacement cost at the beginning and the salvage value at the end of the year is given. If we subtract salvage value from the replacement cost we get the

net value, *i.e.* *C* (capital cost) - *S* (resale value). Rest of the problem is worked as usual.

Years (t) = y	*Running Cost u (y)In Rs.*	*Cumulative Running costM (Y) in Rs.*	*Resale Value S (y)In Rs.*	*C – S (y) In Rs.*	*T (y) = C – S (y) + M (y)In Rs.*	*T (y)/y = G (y) Average cost inRs.*
1	25	25	60	100 – 60 = 40	65	65
2	30	55	50	110 – 50 = 60	115	57.50
3	40	95	40	125 – 40 = 85	180	60
4	50	145	25	140 – 25 = 115	260	65
5	65	210	10	160 – 10 = 150	360	72
6	80	290	0	190 – 0 = 190	480	80

From the table we see that the average annual maintenance cost is minimum at the end of 2nd year and is Rs. 57.50. Hence the machine is to be replaced at the end of 2nd year.

Problem 7.

A fleet owner finds form his past records that the cost per year of running a vehicle whose purchase price is Rs. 50000/- are as under:

Year:	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Running cost in Rs.:	5000	6000	7000	9000	21500	18000	18000
Resale value in Rs.:	30000	15000	7500	3750	2000	2000	2000

Thereafter running cost increases by Rs.2000/- per year but resale value remains constant at Rs. 2000/-. At what stage the replacement is due?

Solution

Years (t) = y	*Running Cost u (y)In Rs.*	*Cumulative Running costM (Y) In Rs.*	*Resale Value S (y)In Rs.*	*C – S (y) In Rs.*	*T (y) = C – S (y) + M (y)In*	*T (y) / y = G (y) Average costin Rs.*

					Rs.	
1	5000	5000	30000	20000	25000	25000
2	6000	11000	15000	35000	46000	23000
3	7000	18000	7500	42500	60500	20166.50
4	9000	27000	3750	46250	73250	**18312.50**
5	21500	48500	2000	48000	96500	19300
6	16000	64500	2000	48000	1,12,500	18750
7	18000	82500	2000	48000	1,30,500	**18642.80**
8	20000	1,02,500	2000	48000	1,50,500	18812.50
9	22000	1,24,500	2000	48000	1,70,500	18944.40

In this problem, the running cost increases from first year (Rs.5000) to Rs. 21, 500 in the 5th year and then it reduces in 6th year and then it increase year wise. Hence there are two minimum annual maintenance cost *i.e.*

Rs. 18,312.50 at the end of 4th year and Rs. 18,642.80 at the end of 7th year. Hence we can conclude that the machine is to be replaced at the end of 4th year. Due to any financial constraint if it is not replaced at the end of 4th year, it must be replaced at the end of 7th year.

Problem 8.

Machine *A* costs Rs. 45,000/- and the operating costs are estimated at Rs. 1000/- for the first year, increasing by Rs. 10,000/- per year in the second and subsequent years. Machine *B* costs Rs.50000/- and operating costs are Rs. 2000/- for the first year, increasing by Rs. 4000/- in the second and subsequent years. If we now have a machine of type *A*, should we replace it by *B*? If so when? Assume both machines have no resale value and future costs are not discounted.

Solution

Let us now calculate the average annual running cost for machine *A* and *B* in the tales given below: (*S* = Rs.0/-)

Machine A

Year (y)	*Running Cost (Rs)u (t)*	*Cumulative Running Cost in Rs, □ u (t) = M (y)*	*Depreciation C – S*	*Total costTC = C – S + M (y)*	*Average costF (y) = TC/ y*
1	1000	1000	45000	46000	46000
2	11000	12000	45000	57000	28500
3	21000	33000	45000	78000	**26000**
4	31000	64000	45000	1,09,000	27200
5	41000	1,05,000	45000	1,50.000	30000
6	51,000	1,56,000	45000	2.01,000	33500

As the annual maintenance cost is minimum at the end of 3rd year, the machine A is to be replaced at the end of 3rd year.

Machine B

Year (y)	*Running Cost (Rs)u (t)*	*Cumulative Running Cost in Rs, □ u (t) = M (y)*	*Depreciation C – S*	*Total costTC = C – S + M (y)*	*Average costF (y) = TC/ y*
1	2000	2000	50000	52000	52000

2	6000	8000	5000 0	58000	29000
3	10000	18000	5000 0	68000	22667
4	14000	32000	5000 0	82000	20500
5	18000	50000	5000 0	1,00,000	**20000**
6	22000	72000	5000 0	1,22,000	20333

As the average annual maintenance cost is minimum at the end of 5th year, the machine is to be replaced at the end of 5th year.

When we compare machine *A* and machine *B*, the average annual maintenance cost of Machine

B is less than that of *B*, the machine *A* should be replaced by machine *A*.

Now to find the time of replacement of machine *A* by machine *B*, the total cost of the machine *A*

in the successive years is computed as given below:

Year:	*1*	*2*	*3*	*4*
Total cost incurred in Rs.:	46000	57000 – 46000	78000 – 57000	1,09,000 – 78000
		= 11000	= 21000	= 31000

The criterion is to replace machine *A* by machine *B* at the age when its running cost for the next year exceeds the lowest average running cost *i.e.* Rs. 20000 per year of machine *B*. From the calculations we can see that running cost of machine is in the third year, *i.e.* Rs. 21000/- is more than the lowest average running cost per year of machine *B* *i.e.* Rs. 20000/- at the end of fifth year. **Hence the machine *A* should be replaced by machine *B* after two years.**

Problem 9.

A taxi owner estimates from his past records that the costs per year for operating taxi whose purchase price when new is Rs.60000/- are as given below:

Age (year):	*1*	*2*	*3*	*4*	*5*
Operating cost in Rs.:	10000	1200 0	15000	1800 0	2000 0

After 5 years, the operating cost is Rs. 6000 × k Where k = 6, 7, 8, 9, 10, *i.e.* 'k' denotes years. If the resale value decreases by 10% of purchase price each year, what is the best replacement policy? Cost of money is zero.

Solution

Capital cost is C = Rs. 60000/-.

Resale value decreases by 10% of capital cost. Hence it reduces by 60000 × (10/100) =Rs. 6000/-

This means $C – S$ increase by Rs. 6000/- every year.

Average annual maintenance cost of machine is:

Year	*Annual Maintenance CostRs. u (t)*	□ *u (t) Rs. M (y)*	*S = Resale Value Rs.*	*C – S Rs.*	*TC = Total CostC – S – M (y)*	*Average annualCost G (y) = TC / y*
1	10000	10000	54000	6000	16000	**16000**
2	12000	22000	48000	12000	34000	17000
3	15000	37000	42000	18000	55000	18333
4	18000	55000	36000	24000	79000	19750
5	20000	75000	30000	30000	1,05,000	21000
6	36000	1,11,000	24000	36000	1,47,000	24500
7	42000	1,53,000	18000	42000	1,95,000	27857
8	48000	2,01,000	12000	48000	2,49,000	31125
9	54000	2,55,000	6000	54000	3,09,000	34333
10	60000	3,15,000	0	60000	3,75,000	37500

As the average annual cost is minimum in the first year itself, the machine is to be replaced every year. We can interpret the situation as: The taxi owner's estimate of operating cost may be wrong or the taxi is of low quality as it is to be replaced every year.

Problem 10

(a) A machine A costs Rs.9000/-. Annual operating costs are Rs. 200/- for the first year and then increases by Rs.2000/- every year. Determine the best age at which the machine *A* is tobe replaced? If the optimum replacement policy is followed, what will be the average yearly cost of owning and operating the machine? Assume machine has no resale value when replaced and that future costs are not discounted.

(b) Machine *B* costs Rs. 10000/-. Annual operating costs are Rs. 400/- for the first year and then increases by Rs. 800/- every year. You have now a machine of type *A*, which is of one year old. Should you replace it with *B*, and if so, when?

Solution

Given that resale value is Rs. zero. Purchase price for machine *A* is Rs. 9000/- and purchase price for machine *B* is Rs. 10,000. Hence for machine *A*, *C* –*S* = Rs. 9000/- and that for *B* is Rs. 10000/-.

Years (y)T	*Annual Maintenance Cost Rs. u (t)*	□ *u (t)Rs. = M (y)*	*C – S Rs.*	*T.C = C – S + M (y) Rs.*	*G (y) = T.C / y Rs.*
1	200	200	9000	9200	9200
2	2200	2400	9000	11400	5700
3	4200	6600	9000	15600	**5200**
4	6200	12800	9000	21800	5450
5	8200	21000	9000	30000	6000

The minimum annual maintenance cost occurs at the end of 3 rd year and it is Rs. 5200/-. Hence the machine *A* is to be replaced at the end of 3 rd year.

Machine B

Years (y)T	*Annual Maintenance Cost Rs. u (t)*	□ *u (t) Rs. = M (y)*	*C – S Rs.*	*T.C = C – S + M (y) Rs.*	*G (y) = T.C / y Rs.*
1	400	400	10000	10400	10400
2	1200	1600	10000	11600	5800
3	2000	3600	10000	13600	4533.33
4	2800	6400	10000	16400	4100
5	3600	10000	10000	20000	**4000**
6	4400	14400	10000	24400	4066.67

As the average annual maintenance cost is minimum *i.e*. Rs. 4000/- at the end of 5 th year, the machine *B* is to be replaced at the end of 5 th year. As the minimum average yearly maintenance cost of machine *B*.

(Rs. 4000/-) is less than that of machine *A i.e*. Rs. 5200, Machine *A* is replaced by machine *B*.

Now we have to workout as when machine *A* is to be replaced by machine *B*? Machine *A* should be replaced when the cost for next year of running this machine becomes more than the average yearlycost for machine *B*.

Total cost of machine *A* in the first year is Rs. 9200/-.

Total cost of machine *A* in the second year is Rs. 11400 – Rs. 9200/- = Rs. 4200/-, (= Total cost of present year – Total cost of previous year) Similarly, the total cost of machine in third year is Rs. 4200/- and in fourth year is Rs.6200/-.

As the cost of running machine *A* in third year (Rs. 4200/-) is more than the average yearly cost for machine *B* (Rs.4000/-), machine *A* should be replaced at the end of second year. Since machine *A* is one year old, it should run for one year more and then it should be replaced.

SIMULATION

INTRODUCTION

Simulation is the most important technique used in analyzing a number of complex systems where the methods discussed in previous chapters are not adequate. There are many real world problems which cannot be represented by a mathematical model due to stochastic nature of the problem, the complexity in problem formulation and many values of the variables are not known in advance and there is no easy way to find these values.

Simulation has become an important tool for tackling the complicated problem of managerial decision-making. Simulation determines the effect of a number of alternate policies without disturbing the real system. Recent advances in simulation methodologies, technical development and software availability have made simulation as one of the most widely and popularly accepted tool in Operation Research. Simulation is a quantitative technique that utilizes a computerized mathematical model in order to represent actual decision-making under conditions of uncertainty for evaluating alternative courses of action based upon facts and assumptions.

John Von Newmann and Stainslaw Ulam made first important application of simulation fordetermining the complicated behaviour of neutrons in a nuclear shielding problem, being too complicated for mathematical analysis. After the remarkable success of this technique on neutron problem, it has become more popular and has many applications in business and industry. The development of digital computers has further increased the rapid progress in the simulation technique. Designers and analysts have long used the techniques of simulation by physical sciences.

Simulation is the representative model of real situation. Fore example, in a city, a children's park withvarious signals and crossing is a simulated model of city traffic. A planetarium is a simulated model of the solar system. In laboratories we perform a number of experiments on simulated model to predict the behaviours of the real system under true environment. For training a pilot, flight simulatorsare used. The simulator under the control of computers gives the same readings as the pilot gets inreal flight. The trainee can intervene when there is signal, like engine failure etc. Simulation is theprocess of generating values using random number without really conducting experiment. Whenever the experiments are costly or infeasible or time-consuming simulation is used to generate the required data.

DEFINITION

1. Simulation is a representation of reality through the use of model or other device, which will react in the same manner as reality under a given set of conditions.
2. Simulation is the use of system model that has the designed characteristic of reality in orderto produce the essence of actual operation.
3. According to Donald G. Malcolm, simulation model may be defined as one which depicts the working of a large scale system of men, machines, materials and information operating over a period of time in a simulated environment of the actual real world conditions.
4. According to Naylor, et al. simulation is a numerical technique for conducting experiments on a digital computer, which involves certain types of mathematical and logical relationships necessary to describe the behaviour and structure of a complex real world system over extended period of time.

There are two types of simulation, they are:

1. Analog Simulation: Simulating the reality in physical form (*e.g.*: Children's park, planetarium,etc.) is known as analog simulation.
2. Computer Simulation: For problems of complex managerial decision-making, the analogue simulation may not be applicable. In such situation, the complex system is formulated into a mathematical model for which a computer programme is developed. Using high-speed computers then solves the problem. Such type of simulation is known as computer simulationor system simulation.

CLASSIFICATION OF SIMULATION MODELS

Simulation models are classified as:

(a) Simulation of Deterministic models:
In the case of these models, the input and output variables are not permitted to be random variables and models are described by exact functional relationship.

(b) Simulation of Probabilistic models:
In such cases method of random sampling is used. The techniques used for solving these models are termed as Monte-Carlo technique.

(c) Simulation of Static Models:
These models do not take variable time into consideration.

(d) Simulation of Dynamic Models:
These models deal with time varying interaction.

ADVANTAGES OF SIMULATION

Simulation is a widely accepted technique of operations research due to the following reasons:

* It is straightforward and flexible.
* It can be used to analyze large and complex real world situations that cannot be solved by conventional quantitative analysis models.

* It is the only method sometimes available.
* It studies the interactive effect of individual components or variables in order to determine which ones are important.
* Simulation model, once constructed, may be used over and again to analyze all kinds of different situations.

* It is the valuable and convenient method of breaking down a complicated system into subsystems and their study. Each of these subsystems works individually or jointly with others.

LIMITATIONS OF SIMULATION TECHNIQUE

* Since simulation model mostly deals with uncertainties, the results of simulation are only reliable approximations involving statistical errors, optimum results cannot be produced by simulation.
* In many situations, it is not possible to identify all the variables, which affect the behaviour of the system.
* In very large and complex problems, it is very difficult to make the computer program in view of the large number of variables and the involved inter–relationship among them.
* For problems requiring the use of computer, simulation may be comparatively costlier and time consuming in many cases.
* Each solution model is unique and its solutions and inferences are not usually transferable to other problems, which can be solved by other techniques.

MONTE-CARLO SIMULATION

The Monte-Carlo method is a simulation technique in which statistical distribution functions are created using a series of random numbers. Working on the digital computer for a few minutes we can create data for months or years. The method is generally used to solve problems which cannot be adequately represented by mathematical models or where solution of the model is not possible by analytical method. Monte-Carlo simulation yields a solution, which should be very close to the optimal, but not necessarily the exact solution. But this technique yields a solution, which converges to the optimal solution as the number of simulated trials tends to infinity. The Monte-Carlo simulation procedure can be summarized in the following steps:

Step 1: Clearly define the problem:

(a) Identify the objectives of the problem.

(b) Identify the main factors, which have the greatest effect on the objective of the problem.

Step 2: Construct an approximate model:

(a) Specify the variables and parameters of the mode.

(b) Formulate the appropriate decision rules, *i.e*. state the conditions under which the experiment is to be performed.

(c) Identify the type of distribution that will be used. Models use either theoretical distributionsor empirical distributions to state the patterns of the occurrence associated with the variables.

(d) Specify the manner in which time will change.

Problem 1.

With the help of a single server queuing model having inter-arrival and service times constantly
1.4 minutes and 3 minutes respectively, explain discrete simulation technique taking 10 minutes as thesimulation period. Find from this average waiting time and percentage of idle time of the facility of a customer. Assume that initially the system is empty and the first customer arrives at time t = 0

Solution

Data: System is initially empty. Service starts as soon as first customer arrives. First customer arrives at t = 0.

The departure time of first customer = 0 + 3 *i.e.* arrival time + service time = 3 minutes (Dep) in the table. The second customer arrives at 1.4 minutes and third arrives at 2.8 minutes (Arr). Until the first customer leaves the system, second and third customers have to wait for service. We can calculatewaiting time for second customer by taking the difference of time of departure of first customer and the time of arrival of second customer *i.e.* 3 – 1.4 = 1.6 minutes. The procedure is shown in the table below:

Time	*Event Arr = arrivalDep = departure*	*Customer Number*	*Waiting time.*
0.0	Arr.	1	--
1.4	Arr.	2	-
2.8	Arr.	3	--
3.0	Dep	1	3.00 – 1.40 = 1.6 min. for customer 2.
4.2	Arr.	4	
5.6	Arr.	5	
6.00	Dep	2	6.00 – 2.80 = 3.2 min. for customer 3.
7.00	Arr.	6	
8.4	Arr.	7	
9.00	Dep.	3	9.00 – 4.20 = 4.8 min. for customer 4
9.80	Arr.	8	

10.00	End of given time	-	10.00 – 5.60 = 4.4 min. for customer 5
			10.00 – 7.00 = 3.0 min. for customer 6
			10.00 – 8.4 = 1.6 min. for customer 7
			10.00 – 9.80 = 0.2 min. for customer 8.

Average waiting time per customer for those who must wait = Sum of waiting time of all customers / number of waiting times taken = (1.4 + 2.8 + 4.2 + 5.6 + 7.0 + 8.4 + 9.8) / 7 = 18.8 / 7 = 2.7 minutes. Percentage of idle time of server = Sum of idle time of server / total time = 0%.

RANDOM NUMBERS

Random number is a number in a sequence of numbers whose probability of occurrence is the same as that of any other number in that sequence.

Pseudo-Random Numbers

Random numbers which are generated by some deterministic process but which satisfy statisticaltest for randomness are called **Pseudo-random numbers.**

Generation of Random Numbers

Using some arithmetic operation one can generate Pseudo-random numbers. These methodsmost commonly specify a procedure, where starting with an initial number called **seed** is generates thesecond number and from that a third number and so on. A number of recursive procedure are available, the most common being the **congruence method** or the **residue method.** This method is described by the expression:

$$\boldsymbol{r_{i+1} = (ar_i + b)\ (\text{modulo } m),}$$

Where a, b and m are constants, r_i and r_{i+1} are the ith and (i + 1)th random numbers.

The expression implies multiplication of a by r_i and addition of b and then division by m. Then r_{i+1} is the remainder or residue. To begin the process of random number generation, in addition to a, b and m, the value of r_0 is also required. It may be any random number and is called **seed**.

Problem 2

With the help of an example explain the additive multiplicative and mixed types of the congruence random number generators.

Solution

The ongruence random number generator is described by the recursive expression

$$\boldsymbol{r_{i+1} = (ar_i + b)\ (\text{modulo } m),}$$

Where a, b and m are constants. The selection of these constants is very important as it determines the starting of random number, which can be obtained by this method. The above expression is for a **mixed type congruential method** as it comprises both multiplication of a and r_i and addition of ar_i and b.

If a = 1, the expression reduces to $r_{i+1} = (r_i + b)$ (modulo m). This is known as **additive type expression.**

When b = 0, the expression obtained is $r_{I+1} = (ar\ I)$ (modulo m), this is known as **multiplicativemethod.**

To illustrate the different types of the congruence methods, let us take a = 16, b = 18 and m = 23 and let the starting random number or seed be r_0 = 1.

(a) Mixed Congruential method: $\boldsymbol{r_{i+1} = (ar_i + b)}$ **(modulo *m*),** therefore,

r_i	***$r_{i+1} = (ar_i + b)$ (modulo m),***	=	*Residue*
r_1	(16 × 1 + 18) / 23	34 / 23	1 residue 11
r_2	(16 × 11 + 18) /23	194 / 23	8 + residue 10
r_3	(16 × 10 + 18) / 23	178 / 23	7 + residue 17
r_4	(16 × 17 + 18) / 23	290 / 23	12 + residue 14
r_5	(16 × 14 + 18) / 23	242 / 23	10 + residue 12
r_6	(16 × 12 + 18) / 23	210 / 23	9 + residue 3
r_7	(16 × 3 + 18) / 23	66 / 23	2 + residue 20
r_8	(16 × 20 + 18) / 23	338 / 23	14 + residue 16
r_9	(16 × 16 + 18) / 23	274 / 23	11 + residue 21
r_{10}	(16 × 21 + 18) / 23	354 / 23	15 + residue 9
r_{11}	(16 × 9 + 18) / 23	162 / 23	6 + residue 1

The random numbers generated by this method are: 1, 11, 10, 17, 14, 12, 3, 20, 16, 21, and 9.

(b) Multiplicative Congruential Method: $r_{i+1} = ar_i$ (modulo m)

r_i	$r_{i+1} = ar_i$ (modulo m)	*Random Number*
r_1	(16 × 1) / 23	0 + Residue 16
r_2	(16 × 16) / 23	11 + Residue 3
r_3	(16 × 3) / 23	2 + Residue 2
r_4	(16 × 2) / 23	1 + Residue 9
r_5	(16 × 9) / 23	6 + Residue 6
r_6	(16 × 6) / 23	4 + Residue 4
r_7	(16 × 4) / 23	2 + Residue 18
r_8	(16 × 18) / 23	12 + Residue 12
r_9	(16 × 12) / 23	8 + Residue 8
r_{10}	(16 × 8) / 23	5 + Residue 13
r_{11}	(16 × 13) / 23	9 + residue 1

The string of random numbers obtained by multiplicative congruential method is 1, 16, 3, 2, 9, 6,
4, 18, 12, 8, and 13.

(c) Additive Congruential Method: $r_{i+1} = (r_i + b)$ **(modulo** m**).**

r_i	$r_{i+1} = (r_i + b)$ **(modulo m)**	*Random Number*
r_1	(1 + 18) / 23	0 + residue 19
r_2	(19 + 18) / 23	1 + residue 14
r_3	(14 + 18) / 23	1 + residue 9
r_4	(9 + 18) / 23	1 + residue 4
r_5	(4 + 18) / 23	0 + residue 22
r_6	(22 + 18) / 23	1 + residue 17
r_7	(17 + 18) / 23	1 + residue 12
r_8	(12 + 18) / 23	1 + residue 7
r_9	(7 + 18) / 23	1 + residue 2
r_{10}	(2 + 18) / 23	0 + residue 20
r_{11}	(20 + 18) / 23	1 + residue 15

The random numbers generated are: 1, 19, 14, 19, 4, 22, 17, 12, 7, 2, 20, and 15.

Problem 3.

	$T = 1$	2	3
The distribution of inter-arrival time in a single server model is	$f(T)$ = ¼	½	¼
	$S = 1$	2	3
And the distribution of service time is	$F(S)$ = ½	¼	¼

Complete the following table using the two digit random numbers as 12, 40, 48, 93, 61, 17, 55, 21, 85, 68 to generate arrivals and 54, 90, 18, 38, 16, 87, 91, 41, 54, 11 to generate the corresponding service times.

Arrival Number	Random Number	Arrival Time	Time Service Begins	Random number	Time Servic ends	Waiting time in Queue

Solution

The distribution of inter-arrival times and the two-digit random numbers assigned to different values of T is as below:

T	*f(T)*	□ *f (T)*	*Random numbers*
1	0.25	0.25	00 to 24
2	0.50	0.75	25 to 74
3	0.25	1.00	75 to 99

Inter-arrival times corresponding to random numbers 12, 40, 48, 93, 61, 17, 55, 21, 85 and 68 are Given 1, 2, 2, 3, 2, 1, 2, 1, 3, 2 respectively. Similarly, the distribution of service times andtwo-digit random numbers assigned to different values of S are as follows:

S	*f (s)*	□ *f (s)*	*Random number*
1	0.50	0.50	00 to 49
2	0.25	0.75	25 to 74
3	0.25	1.00	75 to 99

The simulation is done as follows:

Arrival number	*Random number*	*Arrival time*	*Time Service Begins in Mins.*	*Random number*	*Time Service Ends in Mins.*	*Waiting Time in Queue.*
1	12	1	1	54	3	-
2	40	3	3	90	6	-
3	48	5	6	18	7	1
4	93	8	8	38	9	-
5	61	10	10	16	11	-
6	17	11	11	87	14	-
7	55	13	14	91	17	1
8	21	14	17	41	18	3
9	85	17	18	54	20	1
10	68	19	20	11	21	1

The working of the above table is as below: The simulation of the single-server system starts at zero time. First customer arrives at 1 time unit after that and the service immediately begins. Since the service time for the first customer is 2 time units, service ends at 3 time units. The second customer arrives after an inter-arrival time of 2 time units and goes to service immediately at 3 time units. The third customer who arrives at 5 time units has to wait till the service of 2nd customer ends at 6 units of time. The other entries are also filled on the same logic.

Problem 4.

A coffee house in a busy market operates counter service. The proprietor of the coffee house has approached you with the problem of determining the number of bearers he should employ at the counter. He wants that the average waiting time of the customer should not exceed 2 minutes. After recording the data for a number of days, the following frequency distribution of inter-arrival time of customers and the service time at the counter are established. Simulate the system for 10 arrivals of various alternative number of bearers and determine the suitable answer to the problem.

Inter-arrival time in mins.	*Frequency (%)*	*Service time in mins.*	*Frequency (%)*
0	5	1.0	5
0.5	35	2.0	25
1.0	25	3.0	35
1.5	15	4.0	20
2.0	10	5.0	15
2.5	7		
3.0	3		

Solution

It is queuing situation where customers arrive at counter for taking coffee. Depending upon the number of bearers, the waiting time of the customers will vary. It is like a single queue multi-channel system and waiting customer can enter any of the service channel as and when one becomes available. By taking two-digit random number interarrival and interservice times are as follows:

Random number for arrivals:

Inte-arrival time in minutes	*Frequency*	*Cumulative frequency*	*Random numbers*
0	5	5	00 to 04
0.5	35	40	05 to 39
1.0	25	65	40 to 64
1.5	15`	80	65 to 79
2.0	10	90	80 to 89
2.5	7	97	90 to 96
3.0 `	3	100	97 to 100

Random number for Service:

Service time in minutes	*Frequency*	*Cumulative frequency*	*Random number*
1.0	5	5	00 to 04
2.0	25	30	05 to 29
3.0	35	65	30 to 64
4.0	20	85	65 to 84
5.0	15	100	85 to 99

Arrival Number	*Random Number*	*Inter Arrival Time*	*Random Number*	*Service Time*	*Arrival Time*	*Bearer One Service Begins*	*Bearer One Service Ends*	*Bearer Two Service Begins*	*BearerTwo Service Ends*	*Customer Waiting Time*
1	–	–	31	3	0	0.0	3.0			0
2	48	1.0	46	3	1.0			1.00	4.00	0
3	51	1.0	24	2	2.0	3.0	5.0			1.0
4	06	0.5	54	3	2.5			4.00	7.00	1.5
5	22	0.5	63	3	3.0	5.0	8.00			2.0
6	80	2.01	82	4	5.0			7.00	11.00	2.0
7	56	1.0	32	3	6.0	8.0	11.00			2.0
8	06	0.5	14	2	6.5			11.00	13.00	4.5
9	92	2.5	63	3	9.0	11.0	14.00			2.0
10	51	1.0	18	2	10.0			13.00	15.00	3.0

The customer waiting time with two servers is sometimes greater than 2 minutes. Hence let us try with one more bearers. The table below shows the waiting time of customers with three bearers. With two bearers, total waiting time is 18 minutes. Hence average waiting time is 18 / 10 = 1.8 minutes.

Arrival Number	Server One Service Begins	Server One Service Ends	Server Two Service Begins	Server Two Service Ends	Server Three Service Begins	Service Three Service Ends	Customer Waiting Time
1	0.0	3.0					0
2			1.0	4.0			0
3					2.0	4.0	0
4	3.0	6.0					0.5
5			4.0	7.0			1.0
6					5.0	9.0	0
7	6.0	9.0					0
8			7.0	9.0			0
9					9.0	12.0	0
10	10.0	12.0					0

With three bearers, the total waiting time is 1.5 minutes. Average waiting time is 0.15 minutes.

Similarly, we can also calculate the average waiting time of the bearers.